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FOR THE INCREASE OF COMPLEX MECHANIZATION AND AUTOMATION OF PRODUCTION

During May 12-16 of this year there was held an All-Union Conference of industrial workers on the question of developing complex mechanization and automation of productive processes, and of increasing the output of instruments and means of automation. At this conference, leaders of the party and of the government, and industrial and scientific workers, spoke of the profound successes in mechanization and automation in our country, the blooming of creative force in Soviet science and technology, a striking demonstration of which was given by the launching of the third Soviet sputnik, outfitted with the most modern contrivances of measurement and automation technology.

At this stage of its development, the economy of the Soviet Union has witnessed the maturing of a technological revolution based on automata, atomic energy and radio-electronics. We are entering upon an epoch of creating qualitatively new forms of production which will allow productive work to be increased several times over. In these conditions, complex mechanization and automation of production in the national economy of the USSR is an extraordinarily important economic, political and technological problem.

However, in this most important area there are still grave inadequacies. Despite the resolutions of the twentieth congress of the Communist Party (of the Soviet Union) on the broad automation of a number of branches of industry, this work is proceeding slowly. Theoretical investigations and preparations of technological processes for mechanization and automation, and also the modernization of technological apparatus in view of the requirements of automation,are being performed on a completely inadequate scale. In the production lines of the machine industry there is an inadequate "specific gravity" of semi-automatic and automatic machine stations and automatic continuous-flow lines, many technological processes still requiring the expenditure of physical labor. There is a low level of mechanization of the warehousing and transportation processes. A unifying technological policy is lacking for the cardinal questions of automation and the production of the technological means for mechanization and automation. There is flux and incoherence in tool manufacture, lagging in the areas of creating new methods and apparatus for automatic control and regulation, in particular, transducers for controlling physicochemical processes, electronic controlling machines and other automatic and remote control apparatus, and also a series of machines, mechanisms and other means for complex mechanization.

The theoretical questions constituting the scientific stumbling blocks in automation have been inadequately worked out ;there has been insufficient training of the personnel for the tooling and automation of various branches of industry. There is lagging in economic science when it comes to estimating the economic effectiveness of automation; the attempts at automation have been inadequately generalized, and the state of technical information in these areas is quite poor.

Another serious inadequacy is the slow tempo with which the latest advances are incorporated in the areas of mechanization and automation of production. There is a large hiatus between the time of a scientific discovery and the moment of its realization.

The most important problems in the domain of automata are the following:

1. The creation of automatically working models of production complexes in various branches of the national economy on the basis of new technology and attainments in the techniques of control and regulation, plus the rapid dissemination of the results obtained from such models.

2. The creation of one state-wide system of production of the technical means for automata and remote control, and such a development of industry producing these means which will guarantee the continuous satisfaction of the growing demand for them.

3. The development of research and development organizations and the effective allocation of them in the economic zones in accordance with their features.

Before us the grandiose promises of automation open up: the creation of automated mines, automatic factories, tele-automated productive complexes. With each year automation will acquire ever greater value in every domain of our lives, beginning with the organization of our way of life and ending with space travel.

The problem of our science consists in paving the way toward new solutions to these problems, and in providing ourselves with the means to realize these solutions.

It is not to be doubted that the scientists working in the domains of automata and remote control will do all possible in carrying out the distinctive role which science plays in the matter of complex mechanization and automation, and that they will do their part in the building of Communist society.

THE DETERMINATION OF AN OPTIMAL SYSTEM BY SOME ARBITRARY CRITERION*

V. S. Pugachev

(Moscow)

A method is expounded for the determination of an optimal system, by some arbitrary criterion of the Bayes type, out of the class of all functions to which this criterion can meaningfully be applied, very general assumptions being made as to signals and noise. The solution of the problem of determining the operator of the optimal system reduces to the finding of certain linear operators, and the minimum of some function or functional.

1. Posing of the Problem

The general problem of the optimal systems consists of this, that being given a random function $Z(t)$, $t \in T$, at the input, there must be developed at the output a certain random function $W^*(s)$, which would be as close as possible, in the sense of the criterion chosen, to the random function $W(s)$, $s \in S$. The random function Z is generally the sum of the effective signal plus noise, and the random function W is the result of some transformation of the effective signal.** For brevity, therefore, we shall henceforth call the random function W the effective signal, and the random function Z the observed random function. The random function W^*, which the system must present at its output as the result of reproducing the random function W, we shall call the estimate of signal W.

In various practical problems, different criteria of optimality are chosen, depending on the purpose of the proposed system. All criteria employed in practice can be presented as a minimum condition on the mathematical expectation of some function or functional of the effective signal and its estimate. Thus, the most general criterion possible for an optimum has the form

$$\rho = M\left[r\left(W(s),\ W^*(s)\right)\right] = \min, \tag{1}$$

where r is some function of the signal and its estimate.

This function may depend only on the current value of the signal and its estimate, i.e., be a function in the ordinary sense, or it may depend on all values of the effective signal and its estimate in some region S, i.e., be a functional. If the function r is given in advance, and its form does not depend on the manner in which the estimate W^* of the effective signal is determined, then criterion (1) is a Bayes criterion. Of all the criteria known in practice, only the criterion of minimum loss of information does not fall into the class of Bayes criteria. Theoretically, the criterion of N. I. Andreev [3] is not a Bayes criterion in all cases. However, in all cases in practice when it appears expeditious to use Andreev's criterion, this criterion does fall into the

* The results of this work were presented on December 24, 1957, and January 7 and 21 and February 4, 1958, at the seminar on probabilistic methods of the theory of automatic control at the Institute of Automation and Remote Control of the AN SSSR.

** We will consider here only real random functions and functionals.

class of Bayes criteria. It is thus possible to consider that all optimal criteria arising in practice, with the exception of the criterion of minimum loss of information, are Bayes criteria.

In certain cases, the function $\underline{r}$ contains undetermined parameters which depend on the manner of determining the estimate W^* of the effective signal, it being then necessary so to choose these parameters that certain supplementary conditions are satisfied. Criterion (1), if it contains such a function $\underline{r}$, might be called a conditional Bayes criterion. Among the conditional Bayes criteria is, for example, the well-known Neyman-Pearson criterion for revealing a signal in the presence of noise. The theory of N. I. Andreev shows that the criteria of the class investigated by Andreev are, in the general case, also conditional Bayes criteria.

The correspondence between the observed random function Z and the estimated effective signal W^* is established by the system's operator A:

$$W^*(s) = AZ(t). \tag{2}$$

Consequently, the problem of determining the optimal system in accordance with an arbitrary criterion of type (1) reduces to finding an operator A guaranteeing the minimum mathematical expectation of the function $\underline{r}$.

In many cases, the effective signal presents itself as the sum of a linear combination of several known functions with linear coefficients plus a certain random function. Therefore, the observed random function Z can be given in the form

$$Z(t) = \sum_{h=1}^{N} U_h \varphi_h(t) + X(t), \tag{3}$$

where φ_h are the given functions, U_h are random variables and X is a random function representing the sum of the nonregular portions of the signal and the noise.

The random function W, which is to be reproduced, frequently is the result of some linear transformation of the signal. In such a case it may be represented by the formula

$$W(s) = \sum_{h=1}^{N} U_h \psi_h(s) + Y(s), \tag{4}$$

where the functions ψ_h are the result of subjecting the given functions φ_h to the linear transformation, and the random function Y is the result of applying the same linear transformation to the nonregular portion of the signal.

We give here a method for determining the optimal operator A in accordance with an arbitrary Bayes (or conditional Bayes) criterion of type (1) for the case when the observed random function Z, and the signal to be reproduced W, are represented by Formulas (3) and (4), assuming that the random function vector $[X(t), Y(s)]$ is distributed normally and is statistically independent of the random vector $(U_1, \ldots, U_N)$. Without loss of generality, it may be considered here that the mathematical expectation of the random functions X and Y is identically zero [1, 2].

The arguments $\underline{t}$ and $\underline{s}$ of the random functions will be assumed to be arbitrary scalar to vector variables which, in particular, can have certain discrete components. Therefore, all the theory presented here is applicable both to scalar and to vector random functions. To apply the theory to vector functions, it suffices to consider the components of each vector function as a scalar function of its argument and ordinal number. Thus, the method presented here for determining optimal systems is applicable to both one-dimensional and multidimensional systems.

As is shown by the formula

$$M[r(W, AZ)] = M[M[r(W, AZ) \mid Z]], \tag{5}$$

it suffices for the solution of the posed problem to find an operator A which guarantees a minimum conditional mathematical expectation of the function $\underline{r}$ with respect to the observed random function Z for each possible realization of the random function Z (with the possible exception of certain realizations having zero total probability of occurring):

$$M[r(W, AZ) \mid Z] = \min. \tag{6}$$

Our problem, in its general form, is solved by the method of the canonical decomposition of random functions, which gives a sufficiently simple algorithm for finding the optimal operator.

2. Solution of the Problem For the Criterion of Minimum Mathematical Expectation of a Function of the Current Values of the Effective Signal and its Estimate

We present the vector random function $[X(t), Y(s)]$ by some canonical decomposition [1]:

$$X(t) = \sum_{\nu} V_{\nu} x_{\nu}(t), \quad Y(s) = \sum_{\nu} V_{\nu} y_{\nu}(s), \tag{7}$$

where the V_{ν} are noncorrelated random variables with zero mathematical expectation. As is well known, it is always possible to do this and, moreover, to do it in a nondenumerably infinite number of ways [1]. After the decomposition of (7) has been constructed, we define a system of linear functionals, $\Omega^{(\nu)}$, satisfying, together with the function coordinates x_{ν}, the condition of biorthogonality:

$$\Omega^{(\nu)} x_{\mu} = \delta_{\nu\mu}. \tag{8}$$

This also is always possible to do, using, for example, the methods presented in [1] or in [2]. Having done this we will also have satisfied the condition

$$x_{\nu}(t) = \frac{1}{D_{\nu}} \Omega_u^{(\nu)} K_x(t, u), \tag{9}$$

where D_{ν} are the dispersions of the random variables V_{ν}, and $K_x(t, u)$ is the correlation function of the random function X. The random coefficients of decomposition (7) are expressed by the formula

$$V_{\nu} = \Omega^{(\nu)} X(t). \tag{10}$$

We now determine the random variables $Z_{\nu} = \Omega^{(\nu)} Z$ which, based on (3) and (10), are expressed by the formula

$$Z_{\nu} = \Omega^{(\nu)} Z(t) = D_{\nu} \sum_{h=1}^{N} \alpha_{\nu h} U_h + V_{\nu}, \tag{11}$$

where

$$\alpha_{\nu h} = \frac{1}{D_{\nu}} \Omega^{(\nu)} \varphi_h \quad (h = 1, \ldots, N). \tag{12}$$

Then, taking (7) into account, we obtain

$$\sum_{\nu} Z_{\nu} x_{\nu}(t) = \sum_{h=1}^{N} U_h \sum_{\nu} D_{\nu} \alpha_{\nu h} x_{\nu}(t) + X(t). \tag{13}$$

A comparison of Formulas (3) and (13) shows that in the case when all the functions φ_h can be represented by the decomposition by the coordinate functions

$$\varphi_h(t) = \sum_{\nu} D_{\nu} \alpha_{\nu h} x_{\nu}(t) \quad (h = 1, \ldots, N), \tag{14}$$

then the following equality holds:

$$\sum_{\nu} Z_{\nu} x_{\nu}(t) = Z(t). \tag{15}$$

Thus, in the case when all the functions φ_h can be represented by decomposition (14), the observed random function Z is completely defined by the given random variables Z_ν, and, consequently,

$$M[r(W, AZ) \mid Z] = M[r(W, AZ) \mid Z_\nu]. \tag{16}$$

We assume initially that in the canonical decomposition in (7) of the random function Y there are no random variables V_ν which do not also enter into the decomposition of the random function X. Then, replacing the random variables V_ν in the second formula of (7) by their expression from (11) and substituting the result in (4), we get

$$W(s) = \sum_{h=1}^{N} U_h \omega_h(s) + \sum_\nu Z_\nu y_\nu(s), \tag{17}$$

where

$$\omega_h(s) = \psi_h(s) - \sum_\nu D_\nu \alpha_{\nu h} y_\nu(s) \quad (h = 1, \ldots, N). \tag{18}$$

Thus, giving the joint distribution function of the random variables Z_ν and U_h is equivalent to giving the joint distribution function of the random functions Z and W. In the case when the random function Y has, in decomposition (7), a finite number of random variables V_ν which do not enter into the decomposition of the random function X, then in Formula (17) there appears a linear combination of such random variables V_ν. Obviously, this linear combination may be joined with a linear combination of the random variables U_h. Consequently, the case where the decomposition of the random function Y contains a finite number of random variables V_ν which do not enter into the decomposition of the random function X leads to the previous result with only a simple increase in the number of terms entering into the sum of Formula (4).

We note that a sufficient condition for convergence of the series in (14) and (18) is the convergence of all the series [1, 2]:

$$\sum_\nu D_\nu \alpha_{\nu h}^2 \quad (h = 1, \ldots, N). \tag{19}$$

On the basis of the results presented, the application to the observed random function Z of an arbitrary operator presents itself as a function of the independent variable $\underline{s}$ and the random variables Z_ν, and, consequently, the optimal operator has the form

$$AZ = \mu(s, Z_\nu), \tag{20}$$

where μ is some as yet unknown function. With this, the conditional mathematical expectation of the function $\underline{r}$ with respect to the observed random function is expressed by the formula *

$$M[r(W, AZ) \mid Z_\nu] = \\ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h \omega_h(s) + \sum_\nu Z_\nu y_\nu(s),\ \mu(s, Z_\nu)\right) f(u_h \mid Z_\nu)\, du_1 \ldots du_N, \tag{21}$$

where $f(u_h \mid z_\nu)$ is the conditional probability density of the random vector $(U_1, \ldots, U_N)$ with respect to the random variable Z_ν.

Based on well-known formulas of the theory of probability [1], the conditional probability density of the random variable U_h with respect to the random variable Z_ν is expressed by the formula

$$f(u_h \mid z_\nu) = \frac{f(u_h) f_1(z_\nu \mid u_h)}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_h) f_1(z_\nu \mid u_h)\, du_1 \ldots du_N}, \tag{22}$$

* We denote random variables by capital letters, and the corresponding variables of integration and probability density arguments by lower-case letters.

where $f(u_h)$ is the unconditional probability density of the random variable U_h and $f_1(z_\nu \mid u_h)$ is the conditional probability density of the random variable Z_ν with respect to the variable U_h. In order to find the conditional probability density $f_1(z_\nu \mid u_h)$, we note that the random variables V_ν are distributed normally, since they are the results of a linear transformation (10) of a normally distributed random function X. Consequently, they are not only noncorrelated, but also independent, and their joint probability density equals the product of their probability densities. It follows from this and from Formula (11) that the conditional probability function of the random variable Z_ν with respect to the variable U_h will be normal, and the joint conditional probability density of all the random variables Z_ν with respect to the variable U_h will equal the product of the corresponding conditional probability densities of the individual random variables Z_ν. Taking into account that the random variable V_ν has zero mathematical expectation and dispersion D_ν, we obtain, on the basis of (11), for the conditional probability density of the random variables Z_ν, with respect to the variable U_h the formula

$$f_{z_\nu}(z_\nu \mid u_h) = \frac{1}{\sqrt{2\pi D_\nu}} e^{-\frac{1}{2D_\nu}\left(z_\nu - D_\nu \sum_{h=1}^{N} \alpha_{\nu h} u_h\right)^2}. \tag{23}$$

For the functions f_1 in Formula (22) we substitute the products of Expression (23) corresponding to all values of the index ν, obtaining the following expression for the conditional probability density of the random variables U_h with respect to the random variables Z_ν:

$$f(u_h \mid z_\nu) = p(z_\nu) f(u_h) \exp\left\{\sum_{h=1}^{N} u_h \sum_\nu \alpha_{\nu h} z_\nu - \frac{1}{2} \sum_{p,q=1}^{N} u_p u_q \sum_\nu D_\nu \alpha_{\nu p} \alpha_{\nu q}\right\}, \tag{24}$$

where $p(z_\nu)$ is some function of the variable z_ν, the concrete representation of which is immaterial for our problem.

Substituting Expression (24) in Formula (21), we are led to the conclusion that our problem reduces to determining the function μ for which the integral

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h \omega_h(s) + \sum_\nu z_\nu y_\nu(s),\ \mu(s, z_\nu)\right) f(u_h) \times$$
$$\times \exp\left\{\sum_{h=1}^{N} u_h \sum_\nu \alpha_{\nu h} z_\nu - \frac{1}{2} \sum_{p,q=1}^{N} a_{pq} u_p u_q\right\} du_1 \ldots du_N, \tag{25}$$

where

$$a_{pq} = \sum_\nu D_\nu \alpha_{\nu p} \alpha_{\nu q} \quad (p, q = 1, \ldots, N) \tag{26}$$

has the least possible value for all possible values of the variables $\underline{s}$ and z_ν.

We consider the integral

$$I(s, \eta_0, \eta_1, \ldots, \eta_N, \chi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h \omega_h(s) + \eta_0, \chi\right) f(u_h) \times$$
$$\times \exp\left\{\sum_{h=1}^{N} u_h \eta_h - \frac{1}{2} \sum_{p,q=1}^{N} a_{pq} u_p u_q\right\} du_1 \ldots du_N. \tag{27}$$

For fixed values of $\underline{s}, \eta_0, \eta_1, \ldots, \eta_N$, the integral in (27) is a function of the variable χ. Let χ_0 be the value of the variable χ for which the integral in (27) has the minimum value. It is obvious that χ_0 depends on the values of the variables $\underline{s}, \eta_0, \eta_1, \ldots, \eta_N$:

$$\chi_0 = \chi_0(s, \eta_0, \eta_1, \ldots, \eta_N). \tag{28}$$

Comparing the integrals in (25) and (27), we see that the function $\mu(s, z_\nu)$ for which the integral in (25) assumes its minimum is defined by the formula

$$\mu(s, z_\nu) = \chi_0(s, \eta_0, \eta_1, \dots, \eta_N), \tag{29}$$

where

$$\eta_0 = \sum_\nu z_\nu y_\nu(s), \quad \eta_h = \sum_\nu \alpha_{\nu h} z_\nu \quad (h = 1, \dots, N). \tag{30}$$

If we replace the variable z_ν in Formulas (29) and (30) by the random variable Z_ν, we obtain, on the basis of Formula (20), the following representation for the optimal operator:

$$AZ = \chi_0(s, \mathrm{H}_0, \mathrm{H}_1, \dots, \mathrm{H}_N), \tag{31}$$

where the random variables $\mathrm{H}_0, \mathrm{H}_1, \dots, \mathrm{H}_N$ are linear combinations of the random variables Z_ν:

$$\mathrm{H}_0 = \sum_\nu Z_\nu y_\nu(s) = \sum_\nu y_\nu(s)\, \Omega^{(\nu)} Z(t), \tag{32}$$

$$\mathrm{H}_h = \sum_\nu \alpha_{\nu h} Z_\nu = \sum_\nu \alpha_{\nu h} \Omega^{(\nu)} Z(t) \quad (h = 1, \dots, N). \tag{33}$$

Thus, the optimal operator does not depend directly on the individual random variables Z_ν, but only on $N+1$ linear combinations of them.

Introducing the linear operators

$$A^{(0)} = \sum_\nu y_\nu(s)\, \Omega^{(\nu)}, \quad A^{(h)} = \sum_\nu \alpha_{\nu h} \Omega^{(\nu)} \quad (h = 1, \dots, N), \tag{34}$$

we can rewrite Formulas (32) and (33) in the form

$$\mathrm{H}_h = A^{(h)} Z(t) \quad (h = 0, 1, \dots, N). \tag{35}$$

Formula (31) for the optimal operator then takes the form

$$AZ = \chi_0(s, A^{(0)}Z, A^{(1)}Z, \dots, A^{(N)}Z). \tag{36}$$

Thus, if for any possible values of the variables $\underline{s}, \eta_0, \eta_1, \dots, \eta_N$ there exists a value χ_0 of the variable χ for which the integral $I(s, \eta_0, \eta_1, \dots, \eta_N, \chi)$, considered as a function of χ, attains a minimum, i.e., if for all values of $\underline{s}, \eta_0, \eta_1, \dots, \eta_N, \chi$

$$I(s, \eta_0, \eta_1, \dots, \eta_N, \chi_0) \leqslant I(s, \eta_0, \eta_1, \dots, \eta_N, \chi), \tag{37}$$

then the problem of determining the optimal operator reduces to finding linear operators $A^{(0)}, A^{(1)}, \dots, A^{(N)}$ and a function χ_0, the value of which for given values of $\underline{s}, \eta_0, \eta_1, \dots, \eta_N$ is determined as the value of the variable χ minimizing the integral I, defined by Formula (27). *

---

* In practical problems it may frequently be convenient, in looking for the minima of the integral in (27) and the analogous integral in (58), to make the following change of variable:

$$u = \sum_{h=1}^{N} u_h \omega_h(s) + \eta_0 \tag{A}$$

using $\underline{u}$ instead of one of the variable u_h, for example, instead of u_N (of course, it is necessary for this that $\omega_N(s) \neq 0$, which is always possible to achieve by a proper renumbering of the random variable U_h). Then the integral over the variables $u_1, \dots, u_{N-1}$ can be computed and, setting

(Footnote continued on bottom of following page)

Formulas (34) define the linear operators $A^{(h)}$, in the general case, in the form of infinite series. However, in some particular cases it is possible to find expressions for these operators in finite form. It is easily seen that the linear operators $A^{(h)}$ satisfy the equations

$$A_t^{(0)} K_x(t, u) = K_{yx}(s, u) \quad (u \in T)^{**} \tag{38}$$

$$A_t^{(h)} K_x(t, u) = \varphi_h(u) \quad (u \in T,\ h = 1, \ldots, N). \tag{39}$$

One may convince oneself of this by a direct substitution, taking Formula (14) into account (see also [1, 2]). Therefore, in all cases when it is possible to find solutions to Equations (38) and (39) in finite form, it is possible to avoid recourse to the definition of the operators $A^{(h)}$ by means of the infinite series in (34).

Formulas (18) and (26) determine the functions $\omega_h(s)$ and the quantities a_{pq} in the form of infinite series. These quantities may also be expressed in finite form by means of the operators $A^{(h)}$, if Formulas (12) and (34) are used:

$$\omega_h(s) = \psi_h(s) - A^{(0)}\varphi_h \quad (h = 1, \ldots, N),$$

$$a_{pq} = A^{(p)}\varphi_q = A^{(q)}\varphi_p \quad (p, q = 1, \ldots, N). \tag{40}$$

Thus, although the method presented above for determining the optimal operator was obtained by means of the method of canonical decomposition which, in the most general case, gives an expression for the linear operators $A^{(h)}$ in the form of the infinite series in (34), for the practical application of this method it is possible to avoid the use of canonical decompositions if one knows the solutions of Equations (38) and (39) in finite form.

To estimate the quality of the optimal system, and to compare various real systems with it, it is necessary to be able to compute the mathematical expectation of the function $r(W, W^*)$ for the optimal operator. Since Formula (31) defines the optimal operator as a definite function of a finite number of random variables, this problem is, in principle, quite simply solved. Specifically

$$\rho = M[r(W, AZ)] = M\left[r\left(\sum_{h=1}^{N} U_h\omega_h(s) + \mathrm{H}_0,\ \chi_0(s, \mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_N)\right)\right] =$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_h)\, du_1 \ldots du_N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h(s) + \eta_0,\ \chi_0(s, \eta_0, \eta_1, \ldots, \eta_N)\right) \times$$

$$\times f_2(\eta_0, \eta_1, \ldots, \eta_N \mid u_h)\, d\eta_0 d\eta_1 \ldots d\eta_N, \tag{41}$$

where $f_2(\eta_0, \eta_1, \ldots, \eta_N \mid u_h)$ is the conditional probability density of the random variable H_l with respect to the quantity U_h. Since the random variable H_l is a linear function of the random variables Z_ν, whose conditional probability distribution is normal, the conditional probability distribution of the random variable H_l is also normal, and, consequently,

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$$q(u, \eta_0, \eta_1, \ldots, \eta_N, s) =$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_h) \exp\left\{\sum_{h=1}^{N} u_h\eta_h - \frac{1}{2}\sum_{p,q=1}^{N} a_{pq}u_pu_q\right\} du_1 \ldots du_{N-1}, \tag{B}$$

where the variable u_N should be replaced by its expression from Equation (A), we obtain the integral in (27) in the form

$$I(s, \eta_0, \eta_1, \ldots, \eta_N, \chi) = \int_{-\infty}^{\infty} r(u, \chi)\, q(u, \eta_0, \eta_1, \ldots, \eta_N, s)\, du. \tag{C}$$

**$K_{yx}(s, u)$ denotes the cross-correlation function of the random functions Y and X.

$$f_2(\eta_0, \eta_1, \ldots, \eta_N \mid u_h) =$$

$$= \frac{1}{\sqrt{(2\pi)^{N+1}|K|}} \exp\left\{\frac{1}{2|K|}\begin{vmatrix} 0 & \eta_0 - m_0 \ldots \eta_N - m_N \\ \eta_0 - m_0 & \ldots\ldots\ldots\ldots \\ \ldots\ldots & K \\ \eta_N - m_N & \end{vmatrix}\right\}, \tag{42}$$

where the conditional mathematical expectation m_l and the conditional correlation moment k_{pq} of the random variables H_l are defined, on the basis of the general theory of linear transformations of random functions, by the formulas [1]

$$m_l = M[A^{(l)}Z \mid u_h] = \sum_{h=1}^{N} u_h A^{(l)}\varphi_h = \sum_{h=1}^{N} a_{lh} u_h \quad (l = 0, 1, \ldots, N), \tag{43}$$

$$k_{pq} = M[(A^{(p)}Z - m_p)(A^{(q)}Z - m_q) \mid u_h] =$$

$$= A_t^{(p)} A_{t'}^{(q)} K_x(t, t') \quad (p, q = 0, 1, \ldots, N), \tag{44}$$

and $|K|$ is the determinant of the correlation matrix $K = \|K_{pq}\|$.*

In the special case when $\underline{r}$ is a function of the current value of the difference between the effective signal and its estimate, and the random function Y is identically zero (i.e., when there are no nonregular random components in the effective signal), the results obtained above imply the previously known results obtained by Laning in a more complex heuristic way [6].

The method presented gives an optimal operator only in cases when all the functions φ_h can be represented by a decomposition of type (14) by the coordinate functions x_ν. We now consider the case when certain of the functions φ_h, for example, $\varphi_1, \ldots, \varphi_k$, are not represented by a decomposition of type (14). Then, setting

$$\zeta_h(t) = \varphi_h(t) - \sum_\nu D_\nu \alpha_{\nu h} x_\nu(t) \quad (h = 1, \ldots, k), \tag{45}$$

we obtain from Formula (13)

$$\sum_\nu Z_\nu x_\nu(t) = \sum_{h=1}^{N} U_h \varphi_h(t) + X(t) - \sum_{h=1}^{k} U_h \zeta_h(t) = Z(t) - \sum_{h=1}^{k} U_h \zeta_h(t). \tag{46}$$

Thus, the observed random function Z is completely determined in this case by giving the random variables $U_1, \ldots, U_k$ and all the random variables Z_ν. The random variables Z_ν, in accordance with their definition (11) are, in their turn, completely determined by giving the function Z. We show that, in this case also, the random variables $U_1, \ldots, U_k$ are completely defined by giving the function Z. For this we define the linear functions $\Psi^{(p)}$ which, together with the functions ζ_q, satisfy the biorthogonality condition:

$$\Psi^{(p)}\zeta_q = \delta_{pq} \quad (p, q = 1, \ldots, k). \tag{47}$$

This can always be done, for example, by means of the method presented in [1, 2]. After this we define new linear functionals

$$\Phi^{(p)} = \Psi^{(p)} + \sum_\nu b_{p\nu}\Omega^{(\nu)} \quad (p = 1, \ldots, k) \tag{48}$$

such that they will be orthogonal to all the coordinate functions x_ν:

$$\Phi^{(p)} x_\nu = 0 \quad (p = 1, \ldots, k). \tag{49}$$

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* It follows from Formulas (44) and Equation (39) that $k_{pq} = a_{pq}$ $(p, q = 1, \ldots, N)$.

Since, as a consequence of (12), we have, for all ν,

$$\Omega^{(\nu)}\zeta_h = 0 \quad (h = 1, \ldots, k), \tag{50}$$

the functionals $\Phi^{(p)}$ will also satisfy the biorthogonality condition:

$$\Phi^{(p)}\zeta_q = \delta_{pq} \quad (p, q = 1, \ldots, k). \tag{51}$$

On the basis of (46), (49), and (51), we have

$$0 = \Phi^{(p)}Z(t) - \sum_{n=1}^{k} U_h \Phi^{(p)}\zeta_h = \Phi^{(p)}Z(t) - U_p, \tag{52}$$

whence

$$U_p = \Phi^{(p)}Z(t) \quad (p = 1, \ldots, k), \tag{53}$$

which was to be proved.

Now, in the case being considered, giving the realization of the observed random function Z is completely equivalent to giving the random variables $U_1, \ldots, U_k$ and Z_ν, and

$$\begin{gathered} M[r(W, AZ) \mid Z] = M[r(W, AZ) \mid U_1, \ldots, U_k, Z_\nu] = \\ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{k} U_h\omega_h + \sum_{h=k+1}^{N} u_h\omega_h + \sum_\nu Z_\nu y_\nu, \mu(s, U_1, \ldots, U_k Z_\nu)\right) \times \\ \times f(u_{k+1}, \ldots, u_N \mid U_1, \ldots, U_k, Z_\nu)\, du_{k+1} \ldots du_N, \end{gathered} \tag{54}$$

where $f(u_{k+1}, \ldots, u_N \mid u_1, \ldots, u_k, z_\nu)$ is the conditional probability density of the random variables $U_{k+1}, \ldots, U_N$ with respect to $U_1, \ldots, U_k, Z_\nu$. This conditional probability density is defined by the formula

$$f(u_{k+1}, \ldots, u_N \mid u_1, \ldots, u_k, z_\nu) = \frac{f(u_h) f_1(z_\nu \mid u_h)}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_h) f_1(z_\nu \mid u_h)\, du_{k+1} \ldots du_N}. \tag{55}$$

Thus, using Expression (23) for the conditional probability density of the random variables Z_ν, as before, we obtain

$$\begin{gathered} f(u_{k+1}, \ldots, u_N \mid u_1, \ldots, u_k, z_\nu) = \\ = p(u_1, \ldots, u_k, z_\nu) f(u_h) \exp\left\{\sum_{h=k+1}^{N} u_h \sum_\nu \alpha_{\nu h} z_\nu - \frac{1}{2} \sum_{p,q=1}^{N} a_{pq} u_p u_q\right\}, \end{gathered} \tag{56}$$

where $p(u_1, \ldots, u_k, z_\nu)$ is a function of the variables $u_1, \ldots, u_k, z_\nu$, the concrete form of which is irrelevant to our problem.

By substituting Expression (56) in Formula (54), we come to the conclusion that, to guarantee a minimum condition mathematical expectation of the function r, there must be found a function $\mu(s, u_1, \ldots, u_k, z_\nu)$ for which the integral

$$\begin{gathered} I_1 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h\omega_h(s) + \sum_\nu z_\nu y_\nu(s), \mu(s, u_1, \ldots, u_k, z_\nu)\right) \times \\ \times f(u_h) \exp\left\{\sum_{h=k+1}^{N} u_h \sum_\nu \alpha_{\nu h} z_\nu - \frac{1}{2} \sum_{p,q=1}^{N} a_{pq} u_p u_q\right\} du_{k+1} \ldots du_N \end{gathered} \tag{57}$$

has the minimum value.

Let χ_0 be the value of the variable χ for which the integral

$$I_1(s, \eta_0, \eta_{k+1}, \ldots, \eta_N, u_1, \ldots, u_k, \chi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h \omega_h + \eta_0, \chi\right) \times$$
$$\times f(u_h) \exp\left\{\sum_{h=k+1}^{N} u_h \eta_h - \frac{1}{2} \sum_{p,q=1}^{N} a_{pq} u_p u_q\right\} du_{k+1} \ldots du_N, \tag{58}$$

considered as a function of χ for fixed values of s, η_0, $\eta_{k+1}, \ldots, \eta_N$, $u_1, \ldots, u_k$, has a minimum value. Comparing Formulas (57) and (58), we see that the function μ for which the integral in (57) attains a minimum is defined by the formula

$$\mu(s, u_1, \ldots, u_k, z_\nu) = \chi_0(s, \eta_0, u_1, \ldots, u_k, \eta_{k+1}, \ldots, \eta_N), \tag{59}$$

where the quantities η_l are defined by Formula (30). If we now replace the variables u_h and η_l by the random variables U_h and H_l, and if we take (35) and (53) into account, we obtain the following formula for the optimal operator:

$$AZ = \chi_0(s, A^{(0)}Z, \Phi^{(1)}Z, \ldots, \Phi^{(k)}Z, A^{(k+1)}Z, \ldots, A^{(N)}Z). \tag{60}$$

In the given case, the mathematical expectation of the function r is expressed by the formula

$$\rho = M[r(W, AZ)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_h)\, du_1 \ldots du_N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h \omega_h(s) + \eta_0,\right.$$
$$\left.\chi_0(s, \eta_0, u_1, \ldots, u_k, \eta_{k+1}, \ldots, \eta_N)\right) f_2(\eta_0, \eta_{k+1}, \ldots, \eta_N \mid u_h)\, d\eta_0 d\eta_{k+1} \ldots d\eta_N, \tag{61}$$

where $f_2(\eta_0, \eta_{k+1}, \ldots, \eta_N \mid u_h)$ is the conditional probability density of the random variables H_0, $H_{k+1}, \ldots, H_N$ with respect to the quantity U_h, which is expressed by a formula analogous to (42).

Since, in all the preceding calculations the variable s played the role of a parameter, and the computations of the method presented must be carried out for each given value of s separately, then the theory presented is completely applicable to those cases when the quantities U_h are arbitrary random functions of the variable s. Thanks to this, the theory we have presented is also applicable to those cases in which the canonical decomposition (7) of the random function Y contains an infinite set of random variables V_ν, which do not enter into the canonical decomposition of the random function X. To apply the theory to such cases, it suffices to separate from the canonical decomposition of the random function Y the set of all terms containing variables V_ν which do not enter into the decomposition of the random function X. This sum will be distributed normally and is independent of the sum of the remaining terms in the decomposition of the random function Y. It is therefore possible to denote it by $U_{N+1}(s)$ and thus to transfer it into the order of random variables U_h.

The method given for determining the optimal operator is quickly generalized to the case when the effective signal W is an arbitrary nonlinear function of the random variables U_h and the random function Y. For this it suffices, in all the preceding calculations, to replace the linear function (4) of the variables U_h and Y by an arbitrary nonlinear function of these variables:

$$W(s) = \psi(s, U_1, \ldots, U_N, Y). \tag{62}$$

In the case when the signal W is a vector random function, and the function r depends on all the components of the effective signal and its estimate, the variable χ and the function χ_0 in all the previous calculations are vectors of the same dimensionality. Formula (42) is correspondingly altered to give the proper normal conditional probability density of the random variables H_l. For seeking the minima of the integrals in (27) or (58) in this case, it is possible to apply the method of steepest descent.

3. Solution of the Problem for the Criterion of Minimum Mathematical Expectation of a Functional of the Signal and its Estimate

In the previous section, the region T in which the argument t varied, and in which the random function Z is observed, could depend on the value of the variable s. In that case, the optimal operator minimized the mathematical expectation of the function r for each given value of s. Here, we consider the case when r is a functional on the effective signal $W(s)$ and its estimate $W^*(s)$, defined over a certain region S of variation of s, assuming that the region T does not depend on the variable s, but may depend only on the region S. Moreover, we will consider that the random variables U_h do not depend on s.

It is obvious that, if $Y(s) \equiv 0$, then excluding from the computations of the previous section the variables η_0 and H_0, we can completely extend these computations to the case when r is an arbitrary functional on the effective signal and its estimate. In this case, however, the integrals in (27) and (58) will not depend on s, and will be functionals on the function $\chi(s)$. In this case, the function χ_0 is defined as the function for which the integral in (27) or (58) attains its minimum value.

If the random function Y is not identically zero, then the method of the previous section is not applicable for arbitrary functionals r. In order to extend the method of the previous section to this case, we must limit the class of functionals r. To be precise, we shall consider here only those functionals r which are functionals on some function $\xi(W, W^*)$ of the current values of the effective signal and its estimate:

$$r = r(\xi(W(s), W^*(s))), \tag{63}$$

where the function ξ possesses the property

$$\xi(x, y) = \xi(x + \Delta, \lambda(y, \Delta)) \tag{64}$$

for arbitrary x, y, and Δ. In other words, the function ξ is such that the surface $z = \xi(x, y)$ is formed by advancing a shifting curve, parallel to the xz plane, a certain point of which describes a curve parallel to the xy plane.

In this case

$$r\left(\sum_{h=1}^{N} u_h\omega_h + \eta_0,\ \mu(s, z_v)\right) = r\left(\xi\left(\sum_{h=1}^{N} u_h\omega_h,\ \chi\right)\right), \tag{65}$$

where the quantity χ is defined by the equation

$$\mu(s, z_v) = \lambda(\chi, \eta_0). \tag{66}$$

Substituting Expression (65) in Formula (25), we are led to the conclusion that the problem of determining the function $\mu(s, z_v)$ reduces to the determination of the function $\chi(s) = \chi_0(s, \eta_1, \ldots, \eta_N)$ for which the integral

$$I(\eta_1, \ldots, \eta_N, \chi(s)) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\xi\left(\sum_{h=1}^{N} u_h\omega_h(s), \chi(s)\right)\right) \times$$

$$\times f(u_h) \exp\left\{\sum_{h=1}^{N} u_h\eta_h - \frac{1}{2}\sum_{p,\,q=1}^{N} a_{pq}u_pu_q\right\} du_1 \ldots du_N \tag{67}$$

has a minimum value for the given fixed values of the variables $\eta_1, \ldots, \eta_N$. In solving this problem for all possible values of the variables $\eta_1, \ldots, \eta_N$, and replacing the variables $\eta_0, \ldots, \eta_N$ by the random variables $H_0, \ldots, H_N$, we determine, on the basis of (20) and (66), the optimal operator

$$AZ = \lambda(\chi_0, (s, A^{(1)}Z, \ldots, A^{(N)}Z), A^{(0)}Z). \tag{68}$$

Analogously to the case when the functions $\varphi_1, \ldots, \varphi_k$ are not expressed by the decomposition of (14) by coordinate functions, we shall have, instead of the integral of (67), the integral

$$I_1(u_1, \ldots, u_k, \eta_{k+1}, \ldots, \eta_N, \chi(s)) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\xi\left(\sum_{h=1}^{N} u_h\omega_h(s), \chi(s)\right)\right) \times$$
$$\times f(u_h) \exp\left\{\sum_{h=k+1}^{N} u_h\eta_h - \frac{1}{2}\sum_{p,q=1}^{N} a_{pq}u_pu_q\right\} du_{k+1} \ldots du_N \tag{69}$$

and the optimal operator is determined from the same Formula (68), in which the linear operators $A^{(1)}, \ldots, A^{(k)}$ must be replaced by the linear functionals $\Phi^{(1)}, \ldots, \Phi^{(k)}$.

The mathematical expectation of the functional $\underline{r}$ for the optimal operator is expressed, in this case, by the same Formulas (41) and (61), in which $r\left(\sum_{h=1}^{N} u_h\omega_h + \eta_0, \chi_0\right)$ is replaced by $r\left(\xi\left(\sum_{h=1}^{N} u_h\omega_h, \chi_0\right)\right)$ and, moreover, the variable η_0 is excluded by integrating with respect to it. In accordance with this, the variable η_0 is also excluded from Expression (42) for the conditional probability density of the random variables H_l.

The method presented is employed, in particular, when ξ is an arbitrary linear function of the current value of the effective signal and its estimate.

To find the minima of the functionals in (67) or (69) one may use, for example, the method of steepest descent.

We mention in conclusion that the method presented in this section is applicable only in those cases when, in the decomposition of the random function Y, (7), there are no random variables V_ν which do not also enter into the decomposition of the random function X (or when there are a finite number of such variables).

4. Determining Optimal Operators of Special Forms

In certain cases, it is necessary to find an optimal operator, not in the class of all possible operators, but in a narrower class of operators, for example, in the class of linear operators. Since, as shown in the previous sections, the optimal among all possible operators depends on the random variables Z_ν in depending on the H_l which are linear combinations of them, it is then efficacious, in the given case, to search for the optimal operator of the special type in the form of a definite function of the variables H_l, i.e., in the form

$$AZ = \mu(s, A^{(0)}Z, A^{(1)}Z, \ldots, A^{(N)}Z, \lambda_l), \tag{70}$$

where $\mu(s, \eta_p, \lambda_l)$ is a definite function of the variables $\underline{s}$ and η_p and of the parameters λ_l. In this case, the problem of determining the optimal operator of the given special form reduces to finding the values of the parameters λ_l for which the mathematical expectation of the function $\underline{r}$ has a minimum value. In particular, the optimal linear operator might be sought in the form

$$AZ = \lambda_0A^{(0)}Z + \lambda_1A^{(1)}Z + \cdots + \lambda_NA^{(N)}Z, \tag{71}$$

where the coefficients $\lambda_0, \ldots, \lambda_N$ must be so chosen that the mathematical expectation of the function $\underline{r}$ is a minimum.

If we replace the function χ_0 by the function μ in Formula (41), we obtain the dependence of the mathematical expectation of the function $\underline{r}$ on the parameters λ_l for operators of the form (70):

$$\rho = M[r(W, AZ)] =$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_h)\, du_1 \ldots du_N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h\omega_h + \eta_0, \mu(s, \eta_p, \lambda_l)\right) \times$$
$$\times f_2(\eta_0, \eta_1, \ldots, \eta_N \mid u_h)\, d\eta_0 d\eta_1 \ldots d\eta_N. \tag{72}$$

The problem of determining the optimal operator of the form of (70) then reduces to determining the values of the parameters λ_l for which the quantity ρ, considered as a function of the parameters λ_l, attains a minimum. This problem may be solved numerically by means, for example, of the method of steepest descent.

In cases when the functions $\varphi_1, \ldots, \varphi_k$ cannot be represented by a decomposition of type (14), the operators $A^{(1)}, \ldots, A^{(k)}$ in Formula (70) are replaced by the linear functionals $\Phi^{(1)}, \ldots, \Phi^{(k)}$. With this, the variables $\eta_1, \ldots, \eta_k$ and the integration over them in Formula (72) will be lacking.

5. The Case When the Optimal Operator Can Be Random

If, for arbitrary values of the variables $\eta_0, \eta_1, \ldots, \eta_N$ (or $\eta_0, u_1, \ldots, u_k, \eta_{k+1}, \ldots, \eta_N$), there exists a value χ_0 of the variable χ which minimizes the integral in (27) [or in (58)] of Section 2, or a function $\chi_0(s)$ minimizing the integral in (67) [or in (69)] of Section 3, then this quantity χ_0, or the function $\chi_0(s)$, determines, as we saw, the optimal operator A which possesses such properties that, for an arbitrary realization $\underline{z}$ of the observed random function Z, and for an arbitrary operator B, the following inequality holds:

$$M[r(W, Az) \mid z] \leqslant M[r(W, Bz) \mid z]. \tag{73}$$

It follows from this that in the case when the optimal operator among all possible operators is determined also for arbitrary random operators, the magnitude of the mathematical expectation of the function $\underline{r}$ will not be less than for the operator A determined in Section 2 or Section 3. Thus, if the class of operators among which the optimal operator is to be found is extended by allowing all possible random operators to lie in it, no improvement of the estimate of the effective signal is thereby brought about.

If $\underline{r}$ is a function of the current values of the effective signal and its estimate, and the quantity χ_0, minimizing the integral in (27) or in (58), does not exist for all possible values of $\eta_0, \eta_1, \ldots, \eta_N$ (or, respectively, of $\eta_0, u_1, \ldots, u_k, \eta_{k+1}, \ldots, \eta_N$) but only for some of them, then the use of random operators might improve the estimate of the effective signal.

The problem of determining the optimal random operator leads, in this case, to seeking the random probability density $\delta(\chi, s \mid \eta_0, \eta_1, \ldots, \eta_N)$ which minimizes, for given values of $\eta_0, \eta_1, \ldots, \eta_N$, the mathematical expectation of the integral in (27):

$$\int_{-\infty}^{\infty} I(s, \eta_0, \eta_1, \ldots, \eta_N, \chi)\, \delta(\chi, s \mid \eta_0, \eta_1, \ldots, \eta_N)\, d\chi. \tag{74}$$

In this case, the optimal operator is defined by the formula

$$AZ = \mathrm{X}(s, A^0 Z, A^{(1)} Z, \ldots, A^{(N)} Z), \tag{75}$$

where $\mathrm{X}(s, \eta_0, \eta_1, \ldots, \eta_N)$ is a random function of the variables $\underline{s}, \eta_0, \eta_1, \ldots, \eta_N$, the one-dimensional probability density of which is $\delta(\chi, s \mid \eta_0, \eta_1, \ldots, \eta_N)$.

In the cases where the functions $\varphi_1, \ldots, \varphi_k$ cannot be represented by a decomposition of type (14), the variables $\eta_1, \ldots, \eta_k$ in Formula (74) are replaced by the variables $u_1, \ldots, u_k$, and the linear operators $A^{(1)}, \ldots, A^{(k)}$ are replaced in Formula (75) by the linear functionals $\Phi^{(1)}, \ldots, \Phi^{(k)}$.

6. A Supplement on the Determination of Optimal One-Dimensional and Multidimensional Systems

The general method presented for determining optimal operators has great generality and is applicable to very diverse problems, since the arguments $\underline{t}$ and $\underline{s}$ of the random functions can be arbitrary scalar or vector quantities. In particular, the method presented here is applicable to problems of interpolating and extrapolating scalar and vector random functions of several arguments which arise, for example, in connection with annual forecasts. The method given is also applicable to the design of one-dimensional and multidimensional automatic systems.

If the arguments t and s are different moments of time, then the linear functionals $\Omega^{(\nu)}$ and the linear operators $A^{(h)}$ are expressed, respectively, by the formulas [2]

$$\Omega^{(\nu)}z = \int_T a_\nu(t)\, z(t)\, dt, \tag{76}$$

$$A^{(h)}z = \int_T g^{(h)}(s, t)\, z(t)\, dt \quad (h = 0, 1, \ldots, N), \tag{77}$$

and Equations (38) and (39) become integral equations of the first type:

$$\int_T g^{(0)}(s, t)\, K_x(t, u)\, dt = K_{yx}(s, u) \quad (u \in T), \tag{78}$$

$$\int_T g^{(h)}(s, t)\, K_x(t, u)\, dt = \varphi_h(u) \quad (u \in T;\ h = 1, \ldots, N). \tag{79}$$

The solution of these integral equations, in the most general case, is given by Formulas (34), obtained by the method of canonical decomposition [1, 2]. However, in particular cases, solutions of Equations (78) and (79) may be obtained in finite form by other methods, for example, by the method of integral canonical decomposition [4] or by other well-known methods. Determining the linear operators $A^{(h)}$ and the function χ_0 minimizing the integral in (27) [(58), (67), (69)], we obtain the operator of the desired optimal system from Formula (36) which, in the given case, takes the form

$$AZ = \chi_0\left(s, \int_T g^{(0)}(s, t)\, Z(t)\, dt, \ldots, \int_T g^{(N)}(s, t)\, Z(t)\, dt\right). \tag{80}$$

If t is a set of certain moments of time and the ordinal numbers of the components of the input vector function of the system, and s is a set of certain other moments of time and the ordinal numbers of the output vector function of the system, then the linear functionals $\Omega^{(\nu)}$ and the linear operators $A^{(h)}$ are expressed by the formulas [1, 2]:

$$\Omega^{(\nu)}z = \sum_{q=1}^{n} \int_T a_{\nu q}(t)\, z_q(t)\, dt, \tag{81}$$

$$\{A^{(0)}z\}_p = \sum_{q=1}^{n} \int_T g^{(0)}_{pq}(s, t)\, z_q(t)\, dt \quad (p = 1, \ldots, m)$$

$$A^{(h)}z = \sum_{q=1}^{n} \int_T g^{(h)}_{q}(s, t)\, z_q(t)\, dt \quad (h = 1, \ldots, N), \tag{82}$$

and Equations (38) and (39) take the form of systems of integral equations of the first type:

$$\sum_{q=1}^{n} \int_T g^{(0)}_{pq}(s, t)\, K^x_{ql}(t, u)\, dt = K^{yx}_{pl}(s, u) \quad \begin{pmatrix} u \in T;\ l = 1, \ldots, n, \\ p = 1, \ldots, m \end{pmatrix} \tag{83}$$

$$\sum_{q=1}^{n} \int_T g^{(h)}_{q}(s, t)\, K^x_{ql}(t, u)\, dt = \varphi_{hl}(u) \quad \begin{pmatrix} u \in T;\ l = 1, \ldots, n \\ h = 1, \ldots, N \end{pmatrix}. \tag{84}$$

In the most general case, the solutions of these integral equations are given by Formulas (34), obtained by means of the method of canonical decomposition [1, 2]. Other methods of solving the systems of integral equations in (83) and (84) are still not known. Although there was given, in [4], a general formula for this case, adduced by the method of integral canonical decomposition of random functions, its practical application is complicated by the fact that there still does not exist a method of obtaining the integral canonical decomposition of an arbitrary

random function, particularly a vector function. Determining the linear operators $A^{(h)}$ and the vector function χ_0 minimizing the integral in (27) [(58), (67), (69)], we find the operator of the desired optimal multidimensional system by Formula (36) which, in this case, takes the form

$$\{AZ\}_p = \chi_{0p}\left(s, \sum_{q=1}^{n} \int_T g_{1q}^{(0)}(s, t) Z_q(t)\, dt, \ldots, \sum_{q=1}^{n} \int_T g_q^{(N)}(s, t) Z_q(t)\, dt\right) \quad (p = 1, \ldots, m). \tag{85}$$

Example 1. To determine the optimal system for the extrapolating linear function

$$W(s) = U_1 + U_2 s \tag{86}$$

according to the criterion of a minimum probability that the error exceed a given magnitude a, with the conditions that the random variables U_1 and U_2, and the noise $X(t)$, which is independent of them, all be distributed normally, and that the correlation function of the noise be defined by the formula

$$K_x(t, u) = k_x(t - u) = e^{-c|t-u|}. \tag{87}$$

In the given case, the function r is defined by the formulas

$$r(w, w^*) = \begin{cases} 1 & \text{for } |w - w^*| > a, \\ 0 & \text{for } |w - w^*| < a, \end{cases} \tag{88}$$

and the integral in (27) takes the form

$$I(s, \eta_1, \eta_2, \chi) = C \iint\limits_{|u_1 + u_2 s - \chi| > a} \exp\left\{u_1\eta_1 + u_2\eta_2 - \frac{1}{2}(c_{11} + a_{11})u_1^2 - (c_{12} + a_{12})u_1u_2 - \frac{1}{2}(c_{22} + a_{22})u_2^2\right\} du_1 du_2, \tag{89}$$

where C is some constant and the c_{pq} are elements of the matrix inverse to the correlation matrix of the random vector (U_1, U_2). The integral equations of (79), defining the weight functions of the linear operators $A^{(1)}$ and $A^{(2)}$ in the given case take the form

$$\int_0^T g^{(1)}(s, t)\, e^{-c|t-u|} dt = 1, \qquad \int_0^T g^{(2)}(s, t)\, e^{-c|t-u|} dt = u. \qquad (0 \leqslant u \leqslant T). \tag{90}$$

It is easy to convince oneself, by direct substitution, that the solutions of these integral equations are given by the formulas

$$g^{(1)}(s, t) = \frac{c}{2} + \frac{1}{2}[\delta(t) + \delta(t - T)], \qquad g^{(2)}(s, t) = \frac{c}{2} t - \frac{1}{2c}[\delta(t) - (1 + cT)\delta(t - T)]. \tag{91}$$

It is obvious that the integral in (89) is a quantity proportional to the probability of a random point, governed by a two-dimensional normal distribution, falling in the region defined by the inequality underneath the integral signs. In order that this probability be a minimum, it is necessary so to choose the value of the variable χ that the center of scattering lies in the middle of the strip, i.e., on the line

$$u_1 + u_2 s - \chi_0 = 0 \tag{92}$$

in (u_1, u_2) coordinates. Denoting by b_1 and b_2 the coordinates of the center of scattering, i.e., the values of the variables u_1 and u_2 for which the integrand in (89) is a maximum, we obtain

$$\chi_0 = b_1 + b_2 s. \tag{93}$$

If we equate to zero the partial derivatives of the exponent in (89) for $u_1 = b_1$ and $u_2 = b_2$, we obtain the following linear algebraic equations for the determination of b_1 and b_2:

$$\begin{aligned} (c_{11} + a_{11}) b_1 + (c_{12} + a_{12}) b_2 &= \eta_1, \\ (c_{12} + a_{12}) b_1 + (c_{22} + a_{22}) b_2 &= \eta_2. \end{aligned} \tag{94}$$

The quantities b_1 and b_2 which are defined by these equations are linear functions of the variables η_1 and η_2. Consequently, the quantity χ_0, defined by Formula (93), is also a linear function of η_1 and η_2:

$$\chi_0(s, \eta_1, \eta_2) = \lambda_1\eta_1 + \lambda_2\eta_2, \tag{95}$$

and Formula (80) for the optimal operator takes the form

$$AZ = \int_0^T [\lambda_1 g^{(1)}(s, t) + \lambda_2 g^{(2)}(s, t)]\, Z(t)\, dt, \tag{96}$$

where the weight functions $g^{(1)}$ and $g^{(2)}$ are defined by Formula (91).

In order to compare the operator in (96) with the operator which would be optimal with respect to the criterion of minimum mean square error, we substitute in Formula (95) the expressions from (94) for the quantities η_1 and η_2, and compare the coefficients for b_1 and b_2 with the corresponding coefficients in (93). As a result, we obtain the following linear algebraic equations for determining λ_1 and λ_2:

$$(c_{11} + a_{11})\lambda_1 + (c_{12} + a_{12})\lambda_2 = 1, \quad (c_{12} + a_{12})\lambda_1 + (c_{22} + a_{22})\lambda_2 = s. \tag{97}$$

These equations coincide completely with those which define λ_1 and λ_2 when the criterion of minimum mean square error is used [1, 2]. Thus, the optimal operator in the given case coincides with that obtained by using the criterion of minimum mean square error.

Example 2. With the conditions of the previous example, to find the optimal system using the criterion of minimum mathematical expectation of the function $r(w, w^*) = 1 - e^{-k^2(w-w^*)^2}$, assuming that the random variables U_1 and U_2 are independent, where U_1 is distributed uniformly in the interval $|u_1| < a$, and U_2 is distributed normally.

In this case, the integral in (27) takes the form

$$\begin{aligned} I(s, \eta_1, \eta_2, \chi) = C \int_{-a}^{a} du_1 \int_{-\infty}^{\infty} [1 - e^{-k^2(u_1 + u_2 s - \chi)^2}] \times \\ \times e^{u_1\eta_1 + u_2\eta_2 - \frac{1}{2}a_{11}u_1^2 - a_{12}u_1u_2 - \frac{1}{2}\left(a_{22} + \frac{1}{D}\right)u_2^2} du_1 du_2, \end{aligned} \tag{98}$$

where C is some constant and D is the dispersion of the random variable U_2. To evaluate this integral we apply well-known formulas, obtaining

$$\begin{aligned} I(s, \eta_1, \eta_2, \chi) = \varkappa_0 - \varkappa_1 \Bigg[\Phi\left(a\sqrt{2a} + \sqrt{\frac{2}{a}}\left(\beta_0\chi - \frac{1}{2}\eta_1 - \beta_1\eta_2\right)\right) + \\ + \Phi\left(a\sqrt{2a} - \sqrt{\frac{2}{a}}\left(\beta_0\chi - \frac{1}{2}\eta_1 - \beta_1\eta_2\right)\right)\Bigg] \exp\left\{-\left(\gamma_0 - \frac{\beta_0^2}{a}\right)\chi^2 - \right. \\ \left. - \frac{\beta_0}{a}\chi\eta_1 + \left(\gamma_1 - \frac{2\beta_0\beta_1}{a}\right)\chi\eta_2\right\}, \end{aligned} \tag{99}$$

where κ_0 and κ_1 are functions of $\underline{s}$, η_1 and η_2 which are not essential for our problem,

$$
\begin{gathered}
\alpha = \frac{2k^2 + a_{11} + 2k^2 D\,(a_{11}s^2 - 2a_{12}s + a_{22}) + D\,(a_{11}a_{22} - a_{12}^2)}{2\,(1 + a_{22}D + 2k^2Ds^2)}\,, \\
\beta_0 = k^2\,\frac{1 + a_{22}D - a_{12}Ds}{1 + a_{22}D + 2k^2Ds^2}\,, \qquad \beta_1 = \frac{D}{2}\,\frac{a_{12} + 2k^2 s}{1 + a_{22}D + 2k^2Ds^2}\,, \\
\gamma_0 = k^2\,\frac{1 + a_{22}D}{1 + a_{22}D + 2k^2Ds^2}\,, \qquad \gamma_1 = \frac{2k^2Ds}{1 + a_{22}D + 2k^2Ds^2}\,,
\end{gathered}
\tag{100}
$$

and Φ is the well-known function

$$
\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_0^u e^{-\frac{1}{2}x^2}\,dx. \tag{101}
$$

Designating by ν (p, q, v) the value of $\underline{x}$ for which the function

$$
\omega(x,\ p,\ q,\ v) = e^{-px^2 - vx}\,[\Phi(q + x) + \Phi(q - x)] \tag{102}
$$

assumes a maximum, we determine the value χ_0 of the variable χ for which Expression (99) attains a minimum:

$$
\chi_0 = \frac{1}{2\beta_0}\eta_1 + \frac{\beta_1}{\beta_0}\eta_2 + \frac{\nu}{\beta_0}\sqrt{\frac{\alpha}{2}}\left(\frac{\gamma_0\alpha - \beta_0^2}{2\beta_0},\ a\sqrt{2\alpha},\ \frac{1}{\beta_0^2}\sqrt{\frac{\alpha}{2}}\,[\gamma_0\eta_1 + (2\beta_1\gamma_0 - \beta_0\gamma_1)\eta_2]\right). \tag{103}
$$

Then, Formula (80) for the optimal operator takes the form

$$
AZ = \int_0^T w^{(1)}(s,\ t)\,Z(t)\,dt + \frac{\nu}{\beta_0}\sqrt{\frac{\alpha}{2}}\left(\frac{\gamma_0\alpha - \beta_0^2}{2\beta_0},\ a\sqrt{2\alpha},\ \int_0^T w^{(2)}(s,\ t)\,Z(t)\,dt\right), \tag{104}
$$

where

$$
\begin{gathered}
w^{(1)}(s,\ t) = \frac{1}{2\beta_0}\,g^{(1)}(s,\ t) + \frac{\beta_1}{\beta_0}\,g^{(2)}(s,\ t), \\
w^{(2)}(s,\ t) = \frac{1}{\beta_0^2}\sqrt{\frac{\alpha}{2}}\,[\gamma_0 g^{(1)}(s,\ t) + (2\beta_1\gamma_0 - \beta_0\gamma_1)\,g^{(2)}(s,\ t)].
\end{gathered}
\tag{105}
$$

As we see, in this case the optimal operator is nonlinear.

Example 3. To find the optimal operator for the general case, when $\underline{r}$ is an arbitrary function of the current value of the difference between the effective signal and its estimate, and the random variables U_h in (3) and (4) are distributed normally.

In Example 1, a particular case of the problem under consideration, we found that the optimal operator was linear and coincided with the operator determined by using the criterion of minimum mean square error. If, in Example 2, we had assumed that the variables U_1 and U_2 had a normal joint distribution, then in Example 2, we would also have obtained a linear optimal operator. We now show that, in the general case, for an arbitrary function $\underline{r}$, depending only on the current value of the difference between the effective signal and its estimate, the optimal operator is linear if only the random vector $(U_1, \ldots, U_N)$ is distributed normally.

Let $(m_1, \ldots, m_N)$ be the mathematical expectation of the random vector $(U_1, \ldots, U_N)$, $K = \| k_{pq} \|$ its correlation matrix, and $C = \| c_{pq} \|$ the inverse matrix, $C = K^{-1}$. Then, the probability density of the random vector $(U_1, \ldots, U_N)$ is given by the formula

$$
f(u_\nu) = \sqrt{\frac{|C|}{(2\pi)^N}}\exp\left\{-\frac{1}{2}\sum_{p,q=1}^{N} c_{pq}\,(u_p - m_p)\,(u_q - m_q)\right\}. \tag{106}
$$

where $|C|$ is the determinant of the matrix C, and the integral in (27) takes the form

$$I(s, \eta_0, \eta_1, \ldots, \eta_N, \chi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r\left(\sum_{h=1}^{N} u_h \omega_h(s) + \eta_0 - \chi\right) \times$$

$$\times \sqrt{\frac{|C|}{(2\pi)^N}} \exp\left\{\sum_{k=1}^{N} \eta_h u_h - \frac{1}{2} \sum_{p,q=1}^{N} a_{pq} u_p u_q - \right.$$

$$\left. - \frac{1}{2} \sum_{p,q=1}^{N} c_{pq}(u_p - m_p)(u_q - m_q)\right\} du_1 \ldots du_N. \tag{107}$$

Taking the argument of the function r as a new variable of integration instead of one of the variables $u_1, \ldots, u_N$, we may carry out the integration over the remaining variables u_h by well-known formulas for integrals of exponential functions with quadratic polynomials in the exponents.* As a result, the integral in (107) goes over into the form of a simple single integral. However, in the given case, this transformation is effected much more simply by using purely probabilistic methods. It suffices to note that the integral in (107) is proportional to the mathematical expectation of the random variable r(U), where the random variable U is a linear combination

$$U = \sum_{h=1}^{N} W_h \omega_h(s) + \eta_0 - \chi \tag{108}$$

of the normally distributed random variables $W_1, \ldots, W_N$, the joint probability density of which is defined by the formula

$$f^*(w_1, \ldots, w_N) = C \exp\left\{\sum_{h=1}^{N} \eta_h w_h - \frac{1}{2} \sum_{p,q=1}^{N} a_{pq} w_p w_q - \right.$$

$$\left. - \frac{1}{2} \sum_{p,q=1}^{N} c_{pq}(w_p - m_p)(w_q - m_q)\right\}, \tag{109}$$

where C is a nonessential constant depending on the quantities $\eta_1, \ldots, \eta_N$. Since the variable U is also distributed normally, and its mathematical expectation and dispersion are easily found from the known mathematical expectation and correlation matrix of the random vector $(W_1, \ldots, W_N)$ [1], it is not difficult to write the expression for the probability density of the random variable U, and to express the mathematical expectation of the random variable r(U) by a simple integral.

The mathematical expectations of the random variables $W_1, \ldots, W_N$ can be determined as the values of the variables $w_1, \ldots, w_N$ for which the probability density in (109) is a maximum. As the result, we obtain

$$n_h = M[W_h] = \sum_{p=1}^{N} l_{hp}\left(\eta_p + \sum_{q=1}^{N} c_{pq} m_q\right) \quad (h = 1, \ldots, N), \tag{110}$$

where $\| l_{hp} \|$ is the correlation matrix of the random vector $(W_1, \ldots, W_N)$, inverse to the matrix $\| a_{pq} + c_{pq} \|$. The mathematical expectation and dispersion D of the random variable U, in accordance with well-known formulas in the theory of probability, are defined by the formulas

$$x = M[U] = \sum_{h=1}^{N} n_h \omega_h(s) + \eta_0 - \chi, \quad D = D[U] = \sum_{p,q=1}^{N} l_{pq} \omega_p(s) \omega_q(s). \tag{111}$$

The mathematical expectation of the random variable r(U) and, consequently, the integral in (107), are proportional to the function

$$v(x) = \int_{-\infty}^{\infty} r(u) e^{-\frac{(u-x)^2}{2D}} du. \tag{112}$$

*** See footnote on p. 518.**

This function depends not only on the mathematical expectation of the random variable U, but also on the variable s, of which the dispersion D is a function. However, this dependence is not essential for us and, therefore, we shall not express it in explicit form.

Let x_0 be the value of x for which the integral in (112) attains a minimum. Then, the value χ_0 of the variable χ for which the integral in (112) and, consequently, the integral in (107), attain minima, is determined, based on (111), by the formula

$$\chi_0 = \eta_0 + \sum_{h=1}^{N} \lambda_h \eta_h + \mu, \tag{113}$$

where, for brevity, we have set

$$\lambda_h = \sum_{p=1}^{N} l_{hp} \omega_p(s) \quad (h = 1, \ldots, N). \tag{114}$$

$$\mu = \sum_{h=1}^{N} \lambda_h \sum_{q=1}^{N} c_{hq} m_q - x_0. \tag{115}$$

In accordance with (36), the optimal operator is defined by the formula

$$AZ = A^{(0)}Z + \sum_{h=1}^{N} \lambda_h A^{(h)}Z + \mu \tag{116}$$

We note that since the matrix $\| l_{pq} \|$ is inverse to the matrix $\| a_{pq} + c_{pq} \|$, the quantities λ_h, defined by Formula (114), satisfy the system of linear algebraic equations

$$\sum_{q=1}^{N} (a_{pq} + c_{pq}) \lambda_q = \omega_p(s) \quad (p = 1, \ldots, N). \tag{117}$$

These equations coincide exactly with the equations which define the quantities λ_h in the problem of finding the optimal linear operator in accordance with the criterion of minimum mean square error [1, 2]. Thus, in the given case, the optimal operator is a nonhomogeneous linear operator coinciding exactly, up to a systematic displacement μ, with the linear operator which is optimal when the criterion of minimum mean square error is used with normally distributed effective signal and noise.

We show further that, in the particular case when r is a nondecreasing function of the absolute magnitude of the difference $W - W^*$, the optimal operator we have found completely coincides with that given by using the criterion of minimum mean square error. For this we note that, in the case of an even function r, Formula (112) can be rewritten in the form

$$\nu(x) = \int_0^\infty r(u) \left[e^{-\frac{(u-x)^2}{2D}} + e^{-\frac{(u+x)^2}{2D}} \right] du. \tag{118}$$

Differentiation of this formula gives

$$\nu'(x) = \frac{1}{D} \int_0^\infty r(u) \left[(u - x) e^{-\frac{(u-x)^2}{2D}} - (u + x) e^{-\frac{(u+x)^2}{2D}} \right] du, \tag{119}$$

from which we find

$$\nu'(0) = 0. \tag{120}$$

Differentiating Formula (119), we find the second derivative of the function $\nu(x)$:

$$v''(x) = \frac{1}{D^2}\int_0^\infty r(u)\left\{[(u-x)^2 - D]\, e^{-\frac{(u-x)^2}{2D}} + [(u+x)^2 - D]\, e^{-\frac{(u+x)^2}{2D}}\right\} du, \tag{121}$$

whence

$$v''(0) = \frac{2}{D^2}\int_0^\infty (u^2 - D)\, r(u) e^{-\frac{u^2}{2D}}\, du =$$

$$= \frac{2}{D^2}\left\{\int_{\sqrt{D}}^\infty (u^2 - D)\, r(u)\, e^{-\frac{u^2}{2D}}\, du - \int_0^{\sqrt{D}} (D - u^2)\, r(u)\, e^{-\frac{u^2}{2D}}\, du\right\}. \tag{122}$$

Since

$$\int_{\sqrt{D}}^\infty (u^2 - D) e^{-\frac{u^2}{2D}}\, du = \int_0^{\sqrt{D}} (D - u^2) e^{-\frac{u^2}{2D}}\, du = \sqrt{\frac{D^3}{e}} \tag{123}$$

and the function r is taken to be nondecreasing for $u > 0$, the first integral in (122) is always greater than the second and, consequently

$$v''(0) > 0. \tag{124}$$

Thus, the function ν has a minimum for $x = 0$ for any even function $r(u)$, which is nondecreasing for $u > 0$.

We have thus proved that if the effective signal and the noise are distributed normally, then, for an arbitrary function r depending only on the current value of the difference $W - W^*$, the optimal, by criterion (1), operator of the class of all possible operators for which the criterion is meaningful is a nonhomogeneous linear operator, where only the systematic shift μ of Formula (116) depends on the form of the function r. If the function r is a nondecreasing function of the variables $|W - W^*|$ then the shift μ does not depend on the form of the function r, and is always obtained the same as when the optimal operator is determined by the criterion of minimum mean square error.

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DYNAMICS OF A RELAY-TYPE ELECTRIC SERVOMECHANISM WITH A LOAD VARYING PROPORTIONALLY TO THE MOTION

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The paper treats of the dynamics (combined oscillations) of a relay-type electric servomechanism, the load on which varies proportionally to the motion.

The motion of the servomechanism considered here is described by a complete second-order differential equation, in the right member of which is a relay function with loops and a dead zone.* A complete solution to the problem is given by means of the method of point transformations [1].

1. Introduction

There exists today a rather large number of works, on both the experimental and the theoretical side, in which the auto-oscillations of relay-type servomechanisms are considered.

Among the works dealing with the dynamics of relay-type servomechanisms when the linear portion is described by complete second-order differential equations are works [2-6]. However, these works are primarily given over to the study of the auto-oscillations in the systems considered. In none of them is there to be found a decomposition of parameter space into regions of qualitatively different system behavior.

Moreover, the works cited dealt with system motions described by second-order differential equations, the right sides of which were in the form of the simplest relay function.

The present work investigates the dynamics of electric servomechanisms under certain assumptions which lead to the consideration of nonlinear systems, the motions of which are described by complete second-order differential equations in which the right sides are relay functions, the characteristics of which contain loops and dead zones. The method of point transformations [1] gives a complete solution to the problems of:

1) investigating the structure of the decomposition of phase space by trajectories;

2) finding the decomposition of parameter space into regions of qualitatively different system behavior, i.e., into regions corresponding to the presence or absence of servomechanism auto-oscillation;

3) obtaining analytic expressions for the critical relationships between the servomechanism parameters;

4) investigating the stability of the periodic solutions.

2. Posing the Problem. Equations of Motion of the System.

In practical automatic control problems, widespread use has been made of electrical systems for positioning control by relay-type amplifiers, represented in Fig. 1. Used in this system are a dc motor with independent ex-

* This problem was posed by V. V. Petrov. An approximate determination of the critical relationships between the parameters by means of degenerate limiting cycles was considered in the work of B. N. Petrov, V. V. Petrov, and N. S. Gorskaia, "Investigation of one- and two-stage relay-type servomechanisms with loads varying proportionally to the motion," (Reports of the Institute of Automation and Telemechanics, AN SSSR, 1954).

citation, 1, controlled by a type RP-5 polarized relay, 2. The output shaft is connected to potentiometer slide 4; its deviation α_0 we shall consider equal to zero. Then, the angular discrepancy between the output and input shafts equals the angle of deviation α of the input shaft, taken with reversed sign. Potentiometer slide 5 is connected to the input shaft.

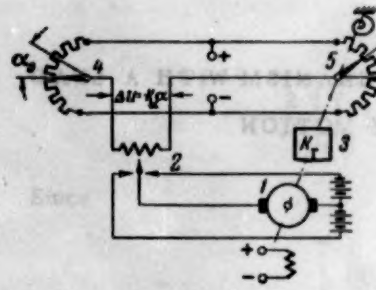

Fig. 1

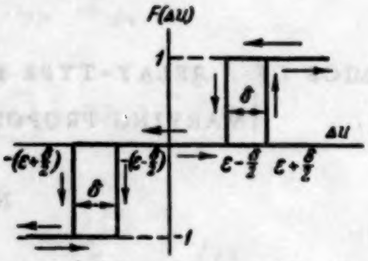

Fig. 2

The electric motor has a load varying proportionally to the angular rotation of the output shaft of reducer 3, i.e., the load moment $M_C = K_C\alpha$.

As is well known, for such a system of an electric motor controlled by a relay amplifier, when the induction of the motor's armature winding can be neglected, the following system of equations is applicable:

discrepancy equation

$$\Delta U = -U_{fb}, \tag{1a}$$

relay element equation

$$U = \Psi[\Delta U] = |U|\,F[\Delta U], \tag{1b}$$

motor electric circuit equation

$$U = IR + E_e, \tag{1c}$$

equations of motion

$$J_{mi}\frac{d\Omega}{dt} = C_M I K_r - K_c\alpha, \tag{1d}$$

$$\Omega = \frac{\Omega_1}{K_r}. \tag{1e}$$

The following notation was used in the equations: I is the armature current, R is the active impedance of the armature winding, U is the voltage impressed on the motor circuit, $E_e = C_e\Omega_1$ is the counter-emf of the motor, C_e and C_M are motor constants, Ω_1 and $\Omega = \frac{d\alpha}{dt}$ are the angular rotational speeds of, respectively, the motor shaft and the reducer's output shaft, K_r is the reduction coefficient, $U_{fb} = K_\alpha\alpha$ is the feedback voltage, proportional to the angular rotation of the input shaft and J_{mi} is the moment of inertia of the reducer's output shaft.

The nonlinear function $F[\Delta U]$, the characteristic of the controlling relay, has the form shown in Fig. 2.

Eliminating from the system of equations (1a) - (1d) all variables except U_{fb}, we obtain

$$A\frac{d^2U_{fb}}{dt^2} + B\frac{dU_{fb}}{dt} + CU_{fb} = -F[U_{fb}], \tag{2}$$

where

$$A = \frac{J_{mi}R}{|U|\,K_\alpha K_r C_M},\quad B = \frac{K_r C_e}{|U|\,K_\alpha C_M},\quad C = \frac{K_c R}{|U|\,K_\alpha K_r C_M}.$$

We introduce the following dimensionless variables: $t^* = \frac{Bt}{A}$, $x = \frac{B^2}{A} U_{fb}$, and also the dimensionless system parameters

$$\mu_\varepsilon = \frac{B^2}{A}\varepsilon,\ \Delta = \frac{B^2}{A}\delta,\ \gamma = \frac{CA}{B^2}.$$

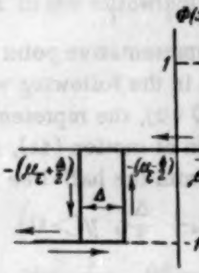

Fig. 3

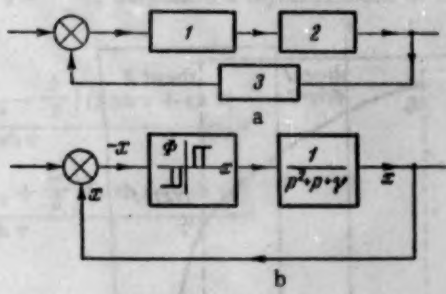

Fig. 4. Servomechanism blockschematic: 1 is the RP-5 controlling relay, 2 is the electric motor and 3 is a rigid feedback connection.

Equation (2), written with dimensionless quantities, takes the following form:

$$\frac{d^2x}{dt^{*2}} + \frac{dx}{dt^*} + \gamma x = -\Phi(x). \tag{3}$$

The nonlinear function $\Phi(x)$ is shown in Fig. 3. This function can be written in the form

$$\Phi(x) = \begin{cases} 1 \text{ for } x \geqslant \mu_\varepsilon - \frac{\Delta}{2}, & \text{if } \Phi(x_0) = 1, \\ 0 \text{ for } |x| < \mu_\varepsilon + \frac{\Delta}{2}, & \text{if } \Phi(x_0) = 0, \\ -1 \text{ for } x \leqslant -\left(\mu_\varepsilon - \frac{\Delta}{2}\right), & \text{if } \Phi(x_0) = -1, \end{cases}$$

where x_0 is the initial condition at the moment of time $t^* = +0$.

Figure 4 gives the block schematic of the electric servomechanism described by Equation (3). Thus, the system is characterized by three parameters, γ, μ_ε, and Δ, the load coefficient, the dead zone, and the loop width in the characteristic of the controlling element.

3. Phase Space

Setting $\dot{x} = y$, we write the equations of motion (3) in the form

$$\dot{x} = y,\ \dot{y} + y + \gamma x = -\Phi(x). \tag{4a}$$

The (x, y) phase space, in accordance with Equations (4a) and the characteristic $\Phi(x)$, consists of three plane sheets, lying one atop the other.

On sheet I the system motion is described by the differential equations

$$\dot{x} = y,\ \dot{y} + y + \gamma x = 0. \tag{4b}$$

Correspondingly, on sheet II

$$\dot{x} = y,\ \dot{y} + y + \gamma x = -1. \tag{4c}$$

The phase plot is symmetric with respect to the origin, since Equations (4b) and (4c) do not change when x and y are both replaced by their negatives.

The singular points of the system of differential equations, (4b) and (4c), have the coordinates (0,0) and $\left(-\frac{1}{\gamma}, 0\right)$ and are stable foci (in the case $\gamma > \frac{1}{4}$) or stable vertices (in the case $\gamma < \frac{1}{4}$). For $(\mu_\epsilon - \frac{\Delta}{2}) > -\frac{1}{\gamma}$, the singular point $\left(-\frac{1}{\gamma}, 0\right)$ does not lie in its region of definition.

We investigate the actual practical case when

$$\left|\mu_\epsilon \pm \frac{\Delta}{2}\right| < \frac{1}{\gamma}.$$

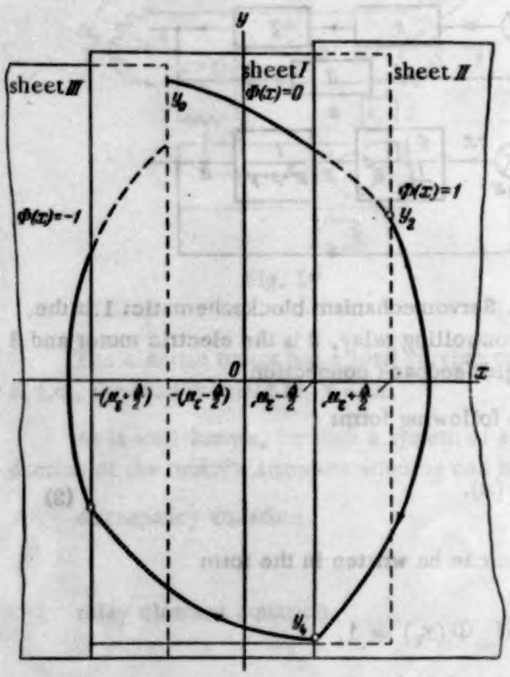

Fig. 5

The motion of the representative point on the three-sheeted phase space occurs in the following way (Fig. 5). In moving on sheet I [$\Phi(x) = 0$], the representative point, in accordance with the equation of motion (4a), moves during a certain interval of time from the half-line

$$\left(x = -\mu_\epsilon + \frac{\Delta}{2},\ y > 0\right)$$

to the half-line

$$\left(x = \mu_\epsilon + \frac{\Delta}{2},\ y \geqslant 0\right).$$

It then moves from some point on this line

$$\left(x = \mu_\epsilon + \frac{\Delta}{2},\ y \geqslant 0\right)$$

now on sheet II [$\Phi(x) = 1$], falling on half-line

$$\left(x = \mu_\epsilon - \frac{\Delta}{2},\ y < 0\right)$$

at some moment of time, then once again moves to sheet I, still does not reach the line

$$\left(x = -\mu_\epsilon - \frac{\Delta}{2},\ y < 0\right),$$

transfers over to sheet III [$\Phi(x) = -1$], etc.

It may happen that by its motion on the three-sheeted phase space the representative point describes a closed curve, a limiting cycle, mirroring the presence in the system of a periodic motion.

4. Point Transformations

It follows from our consideration of the structure of phase space and the motion of the representative point on it that the problem of discovering limiting cycles, and the investigation of the decomposition of phase space by trajectories, lead to the investigation of the point transformations of lines into lines. Each such transformation consists of two point transformations, or of the corresponding functions: of the function (A), which corresponds to the transfer of the representative point on sheet I of the phase space, determining the law transforming the half-line $\left(x = -\mu_\epsilon + \frac{\Delta}{2},\ y > 0\right)$ into the half-line $\left(x = \mu_\epsilon + \frac{\Delta}{2},\ y \geqslant 0\right)$, and of the function (B) which corresponds to the transfer of the representative point on sheet II of the phase space, determining the law transforming the half line $\left(x = \mu_\epsilon + \frac{\Delta}{2},\ y \geqslant 0\right)$ into the half line $\left(x = \mu_\epsilon - \frac{\Delta}{2},\ y < 0\right)$.

With such a representation, a fixed point of these half-transformations will, obviously, attest to the presence of a limiting cycle on the phase space. We find expressions for the corresponding functions.

<u>Sheet I [$\Phi(x) = 0$] of Phase Space.</u> At the moment of time $t^* = 0$, let the representative point (Fig. 5) start its motion from the point $\left(x = -\mu_\epsilon + \frac{\Delta}{2},\ y = y_0\right)$. With these initial conditions we determine the constants of integration in the solution of the system of differential equations (4a) for $\Phi(x) = 0$. If, after the interval of time $t^* = \frac{\tau}{w}$, where $w^2 = \frac{1}{4} - \gamma > 0$, the representative point arrives at the point $\left(x = \mu_\epsilon + \frac{\Delta}{2},\ y = y_1\right)$, we then obtain from the solution we have found*

$$\mu_\varepsilon + \frac{\Delta}{2} = \frac{1}{w} e^{-\frac{\tau}{2w}} \left\{ \left[y_0 + \frac{1}{2}\left(-\mu_\varepsilon + \frac{\Delta}{2} \right) \right] \operatorname{sh} \tau + w\left(-\mu_\varepsilon + \frac{\Delta}{2} \right) \operatorname{ch} \tau \right\},$$

$$y_2 = \frac{1}{w} e^{-\frac{\tau}{2w}} \left\{ \left[-\frac{1}{2} y_0 - \left(-\mu_\varepsilon + \frac{\Delta}{2} \right)\left(\frac{1}{4} - w^2 \right) \right] \operatorname{sh} \tau + w y_0 \operatorname{ch} \tau \right\}. \tag{5}$$

Solving these equations with respect to $u_1 = \frac{y_0}{w}$ and $v = \frac{y_2}{w}$, we obtain a representation of the desired function (A) in the following form:

$$u_1 = \frac{\left(\mu_\varepsilon + \frac{\Delta}{2} \right) e^{p\tau} + \left(\mu_\varepsilon - \frac{\Delta}{2} \right) (p \operatorname{sh} \tau + \operatorname{ch} \tau)}{\operatorname{sh} \tau}, \tag{6}$$

$$v = \frac{\left(\mu_\varepsilon - \frac{\Delta}{2} \right) e^{-p\tau} - \left(\mu_\varepsilon + \frac{\Delta}{2} \right) (p \operatorname{sh} \tau - \operatorname{ch} \tau)}{\operatorname{sh} \tau}.$$

Here $p = \frac{1}{2w}$, $w^2 = \frac{1}{4} - \gamma > 0$, $t^* = \frac{\tau}{w}$.

In the case $\gamma > \frac{1}{4}$ we find, analogously,

$$\mu_\varepsilon + \frac{\Delta}{2} = \frac{1}{w_1} e^{\frac{-\tau}{2w_1}} \left\{ \left[y_0 + \frac{1}{2}\left(-\mu_\varepsilon + \frac{\Delta}{2} \right) \right] \sin \tau + w_1\left(-\mu_\varepsilon + \frac{\Delta}{2} \right) \cos \tau \right\},$$

$$y_2 = \frac{1}{w_1} e^{\frac{-\tau}{2w_1}} \left\{ \left[-\frac{1}{2} y_0 - \left(-\mu_\varepsilon + \frac{\Delta}{2} \right)\left(\frac{1}{4} - w_1^2 \right) \right] \sin \tau + w_1 y_0 \cos \tau \right\}. \tag{7}$$

Solving these equations for $u_1 = \frac{y_0}{w_1}$ and $v = \frac{y_2}{w_1}$, we find the expressions for the desired function (A) in the following form:

$$u_1 = \frac{1}{\sin \tau} \left[\left(\mu_\varepsilon + \frac{\Delta}{2} \right) e^{p\tau} + \left(\mu_\varepsilon - \frac{\Delta}{2} \right) (p \sin \tau + \cos \tau) \right], \tag{8}$$

$$v = \frac{1}{\sin \tau} \left[\left(\mu_\varepsilon - \frac{\Delta}{2} \right) e^{-p\tau} - \left(\mu_\varepsilon + \frac{\Delta}{2} \right) (p \sin \tau - \cos \tau) \right].$$

Here $p = \frac{1}{2w_1}$, $w_1^2 = \gamma - \frac{1}{4} > 0$, $t^* = \frac{\tau}{w_1}$.

<u>Sheet II [Φ (x) = 1] of Phase Space.</u> We assume that at the moment of time $t^* = 0$, the representative point (Fig. 5) lies on the half-line $\left(x = \mu_\varepsilon + \frac{\Delta}{2},\ y \geqslant 0 \right)$ at the point $\left(x = \mu_\varepsilon + \frac{\Delta}{2},\ y = y_2,\ y_2 > 0 \right)$. With these initial conditions we determine the constants of integration in the solution of the system of differential equations (4a) for $\Phi(x) = 1$.

If, after the interval of time $t^* = \frac{\tau_1}{w}$, where $w^2 = \frac{1}{4} - \gamma > 0$, the representative point is found at the point $x = \mu_\varepsilon - \frac{\Delta}{2}$, $y = y_4 = -|y_4|$, then from the solution we have obtained we find

$$\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma} = \frac{1}{w} e^{\frac{-\tau_1}{2w}} \left\{ \left[y_2 + \frac{1}{2}\left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma} \right) \right] \operatorname{sh} \tau_1 + w\left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma} \right) \operatorname{ch} \tau_1 \right\},$$

$$|y_4| = \frac{1}{w} e^{\frac{-\tau_1}{2w}} \left\{ \left[\frac{1}{2} y_2 + \left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma} \right) \right]\left(\frac{1}{4} - w^2 \right) \operatorname{sh} \tau_1 - w y_2 \operatorname{ch} \tau_1 \right\}. \tag{9}$$

• 'sh' ≡ 'sinh'; 'ch' ≡ 'cosh'; 'ctg' ≡ 'cot'; 'tg' ≡ 'tan' – Publisher.

Solving these equations for $u_2 = \frac{|y_4|}{w}$ and $v = \frac{y_2}{w}$, we obtain a representation of the desired function (B) in the following form:

$$u_2 = \frac{1}{\operatorname{sh}\tau_1}\left[\left(\mu_* + \frac{\Delta}{2} + \frac{1}{\gamma}\right)e^{-p\tau_1} + \left(\mu_* - \frac{\Delta}{2} + \frac{1}{\gamma}\right)(p\operatorname{sh}\tau_1 - \operatorname{ch}\tau_1)\right],$$
$$v = \frac{1}{\operatorname{sh}\tau_1}\left[\left(\mu_* - \frac{\Delta}{2} + \frac{1}{\gamma}\right)e^{p\tau_1} - \left(\mu_* + \frac{\Delta}{2} + \frac{1}{\gamma}\right)(p\operatorname{sh}\tau_1 + \operatorname{ch}\tau_1)\right]. \tag{10}$$

Here $p = \frac{1}{2w}$, $w^2 = \frac{1}{4} - \gamma > 0$, $t^* = \frac{\tau_1}{w}$.

In the case $\gamma < \frac{1}{4}$ we find, analogously,

$$\mu_* - \frac{\Delta}{2} + \frac{1}{\gamma} = \frac{1}{w_1}e^{\frac{-\tau_1}{2w_1}}\left\{\left[y_2 + \frac{1}{2}\left(\mu_* + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\right]\sin\tau_1 + \right.$$
$$\left. + w_1\left(\mu_* + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\cos\tau_1\right\},$$
$$|y_4| = \frac{1}{w_1}e^{\frac{-\tau_1}{2w_1}}\left\{\left[\frac{1}{2}y_2 + \left(\mu_* + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\right]\left(\frac{1}{4} - w_1^2\right)\sin\tau_1 - w_1 y_2\cos\tau_1\right\}. \tag{11}$$

Solving these equations for $u_2 = \frac{|y_4|}{w_1}$ and $v = \frac{y_2}{w_1}$, we find an expression for the desired function (B) in the following form:

$$u_2 = \frac{1}{\sin\tau_1}\left[\left(\mu_* + \frac{\Delta}{2} + \frac{1}{\gamma}\right)e^{-p\tau_1} + \left(\mu_* - \frac{\Delta}{2} - \frac{1}{\gamma}\right)(p\sin\tau_1 - \cos\tau_1)\right],$$
$$v = \frac{1}{\sin\tau_1}\left[\left(\mu_* - \frac{\Delta}{2} + \frac{1}{\gamma}\right)e^{p\tau_1} - \left(\mu_* + \frac{\Delta}{2} + \frac{1}{\gamma}\right)(p\sin\tau_1 + \cos\tau_1)\right]. \tag{12}$$

Here $p = \frac{1}{2w_1}$, $w_1^2 = \gamma - \frac{1}{4}$, $t^* = \frac{\tau_1}{w_1}$.

The curves given parametrically by Equations (6) [or (8)] and (10) [or (12)] are defined for all positive values of τ and τ_1. However, because of the rationale for introducing the point transformations, the following rules of "selection" [8] must be satisfied by the branches to be retained out of the set of all curves which could be constructed in the first quadrant when τ and τ_1 vary from zero to infinity:

1) for transformation (A), for each value $u_1 > 0$, that value of $v > 0$ is chosen which is defined by the least root τ of the first equation of System (6) [or (8)];

2) for transformation (B), for each value $v > 0$, that value of $u_2 > 0$ is chosen which is defined by the least root τ_1 of the second equation of System (10) [or (12)].

The curves thus singled out for (A) and (B) are given geometrically by the functions $u_1 = u_1(v)$ and $u_2 = u_2(v)$, forming the corresponding point transformations (A) and (B). These curves, constructed in the Cartesian coordinate system, where the quantity $v > 0$ is laid out along the axis of abscissas and the quantities $u_1 > 0$ and $u_2 > 0$ are laid out along the axis of ordinates, comprise the so-called (u_1, u_2, v)-diagram.

An iteration of the point transformations (A) and (B) here corresponds to a broken line, consisting of line segments parallel to the coordinate axes, inscribed between the curves for (A) and (B). An intersection of the curves on the (u_1, u_2, v)-diagram bespeaks the presence of a fixed point of the half-transformation, i.e., the presence of a limiting cycle on the phase space, and the character of the intersection of these curves is indicative of the stability or instability of this limiting cycle.*

* The criterion for stability (in the narrow sense) of a limiting cycle is that, for the point of intersection of the curves $u_1 = u_1(v)$ and $u_2 = u_2(v)$ corresponding to this limiting cycle, the inequality $\frac{du_2/dv}{du_1/dv} < 1$ hold. With this, the angle of slope for the tangent may be either positive or negative.

Below we give various cases which can be treated with the transformations (A) and (B) which we have obtained.

For the case $\gamma < \frac{1}{4}$ the function corresponding to (A) is found from (6) and the function corresponding to (B) is found from (10). We investigate the behavior of the curves $u_1 = u_1(\tau)$, $v = v(\tau)$, and $u_2 = u_2(\tau_1)$ $v = v(\tau_1)$, defined by these expressions [8]. For this, we consider the qualitative behavior of Curve (6). The expressions for the derivatives may be written in the form

$$\frac{du_1}{d\tau} = -\frac{v}{\operatorname{sh}\tau} e^{p\tau}, \quad \frac{dv}{d\tau} = -\frac{u_1}{\operatorname{sh}\tau} e^{-p\tau} \tag{13}$$

Consequently,

$$\frac{du_1}{dv} = \frac{v}{u_1} e^{2p\tau},$$

$$\frac{d^2u_1}{dv^2} = \frac{2(p^2-1)e^{3p\tau}}{u_1^3}\left[\left(\mu_\epsilon^2 + \frac{\Delta^2}{4}\right)\operatorname{sh} p\tau + \left(\mu_\epsilon^2 - \frac{\Delta^2}{4}\right) p \operatorname{sh}\tau + \mu_\epsilon \Delta \operatorname{ch} p\tau\right]. \tag{14}$$

From this it is easy to deduce the properties stemming from these expressions for the function and its derivatives. It follows from (14) that the derivatives $\frac{du_1}{dv}$ and $\frac{d^2u_1}{dv^2}$ in the first quadrant are always positive as τ varies from zero to infinity and, consequently, the curve $u_1 = u_1(v)$ remains concave as τ varies within the stated limits. For $\tau \to 0$, the derivative du_1/dv tends to unity. The curve $u_1 = u_1(v)$ has the asymptote $u_1 = v + 4p\mu_\epsilon$.

For small τ we have

$$u_1 = 2\frac{\mu_\epsilon}{\tau} + \mu_\epsilon p + \left(\mu_\epsilon + \frac{\Delta}{2}\right)\left[p^2\frac{\tau}{2} + p^3\frac{\tau^2}{3!} + \ldots\right],$$

$$v = 2\frac{\mu_\epsilon}{\tau} - \mu_\epsilon p + \left(\mu_\epsilon - \frac{\Delta}{2}\right)\left[p^2\frac{\tau}{2} - p^3\frac{\tau^2}{3!} + \ldots\right],$$

and, for $\tau \to 0$, the functions $u_1 \to \infty$ and $v \to \infty$.

For $v = 0$, the derivative $\frac{du_1}{dv}$ equals zero, and in the interval $0 < v < \infty$ the inequality $1 > \frac{du_1}{dv} > 0$ holds.

Thus, as τ decreases from some value $\bar{\tau}$, for which $\frac{du_1}{dv} = 0$, to zero, Curve (6) enters the first quadrant from some point on the axis of ordinates and thereafter approximates to the asymptote $u_1 = v + 4p\mu_\epsilon$, remaining always concave.

We now consider the qualitative behavior of the curve $u_2 = u_2(v)$, defined by Expression (10). The expressions for the derivatives have the form

$$\frac{du_2}{d\tau_1} = \frac{v}{\operatorname{sh}\tau_1} e^{-p\tau_1}, \quad \frac{dv}{d\tau_1} = \frac{u_2}{\operatorname{sh}\tau_1} e^{p\tau_1}. \tag{15}$$

Consequently,

$$\frac{du_2}{dv} = \frac{v}{u_2} e^{-2p\tau_1},$$

$$\frac{d^2u_2}{dv^2} = \frac{2(p^2-1)e^{-3p\tau_1}}{u_2^3}\left\{-\left[\left(\mu_\epsilon + \frac{1}{\gamma}\right)^2 + \frac{\Delta^2}{4}\right]\operatorname{sh} p\tau_1 + \right. \tag{16}$$

$$\left. + \left[\left(\mu_\epsilon + \frac{1}{\gamma}\right)^2 - \frac{\Delta^2}{4}\right] p \operatorname{sh}\tau_1 + \left(\mu_\epsilon + \frac{1}{\gamma}\right)\Delta \operatorname{ch} p\tau_1\right\}.$$

It follows from (16) that, in the first quadrant, the derivative $\frac{du_2}{dv}$ is always positive as τ_1 varies from

zero to infinity. The derivative $\frac{d^2u_2}{dv^2}$ can be either negative or positive, depending on the value of τ_1, i.e., Curve (10) has a point of inflection.* For small τ_1 we have

$$u_2 = \frac{\Delta}{\tau_2} - \Delta p + \left(\mu_\epsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\left[p^2 \frac{\tau_1}{2} - p^3 \frac{\tau_1^2}{3!} + \ldots\right],$$

$$v = -\frac{\Delta}{\tau_1} - \Delta p + \left(\mu_\epsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right)\left[p^2 \frac{\tau_1}{2} + p^3 \frac{\tau_1^2}{3!} + \ldots\right],$$

and for $\tau_1 \to \infty$, the functions $u_2 \to \left(\mu_\epsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right)(p-1)$, $v \to \infty$.

For some value $\tau_1 = \bar{\tau}_1$, the function $v = 0$ and the derivative $\frac{du_2}{dv} = 0$. For $\tau_1 > \bar{\tau}_1$, the function $v > 0$ and the derivative $\frac{du_2}{dv} > 0$, since $\frac{dv}{d\tau_1} > 0$ and the numerator in $\frac{du_2}{dv}$ increases with increasing τ_1. As τ_1 tends to infinity, the derivative $\frac{du_2}{dv}$ tends to zero. For $v > 0$, the derivative $\frac{du_2}{dv} < 1$.

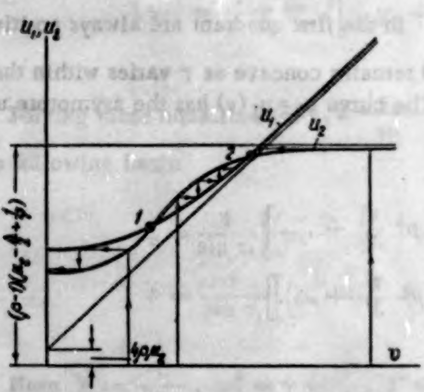

Fig. 6. Diagram of the point transformation. Point 1 corresponds to an unstable limiting cycle, and point 2 to a stable limiting cycle.

Thus, the function $u_2 = u_2(v) > 0$, for $v = 0$, continues to increase up to the limit $(p-1)\left(\mu_\epsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right)$ with increasing $\underline{v}$.

The derivative $\frac{du_2}{dv} = 0$ for $v = 0$ and for $v = \infty$. Within the interval $0 < v < \infty$ the inequality $1 > \frac{du_2}{dv} > 0$ holds.

Thus, as τ_1 increases from zero to infinity, Curve (10) enters into the first quadrant from some point on the axis of ordinates and thereafter approximates to the asymptote

$$u_2 = (p-1)\left(\mu_\epsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right).$$

Figure 6 shows the diagram of the point transformation formed by the curves $u_1 = u_1(v)$ and $u_2 = u_2(v)$ which we have found. The qualitative investigation, carried out above, of the behavior of these curves shows that Curves (6) and (10) may have one or two points of intersection to within an even number of points (i.e., $1 + 2k$ or $2 + 2k$, where $k = 1, 2, 3, \ldots$). It is also possible that there are no points of intersection.

If these curves have two points of intersection, then on the phase space there exists one unstable and one stable limiting cycle. These two cycles may fuse into one semistable limiting cycle (Fig. 7, a) and thereafter disappear. The case is also possible when only one of the limiting cycles (the unstable one) disappears, and the other cycle persists (Fig. 7, b).

The boundary in $(\gamma, \Delta, \mu_\epsilon)$ space corresponding to this plot (a bifurcating parameter ratio) will separate the region of parameters for which two cycles exist from the region of parameters for which one cycle exists.

The curves $u_1 = u_1(v)$ and $u_2 = u_2(v)$ satisfy, respectively, the equations

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* Indeed, for small τ_1 the derivative $\frac{d^2u_2}{dv^2}$ is positive [Curve (10) is concave], since the expression in curly brackets in Equation (16) is positive, having the form $\left\{\left(\mu_\epsilon + \frac{1}{\gamma}\right)\Delta - \frac{\Delta^2}{2} p\tau_1\right\}$ for small τ_1.

$$\frac{du_1}{dv} = \frac{v}{u_1} e^{2p\tau}, \quad \frac{du_2}{dv} = \frac{v}{u_2} e^{-2p\tau_1}.$$

At the point of contact of these curves, $u_1(\tau) = u_2(\tau_1)$, $v(\tau) = v(\tau_1)$.

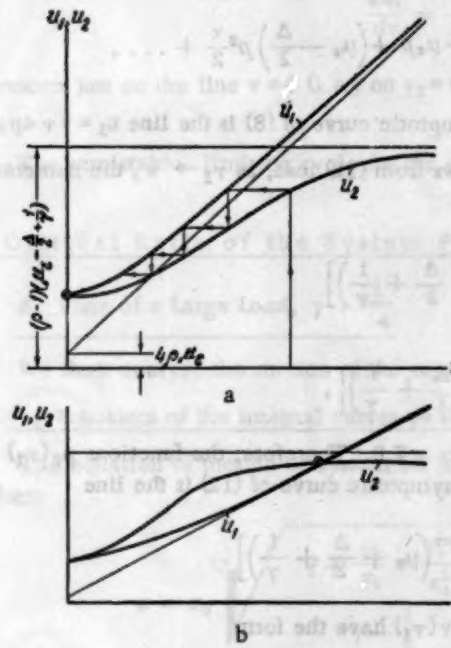

Fig. 7. Diagram of point transformations: a) is for the case of critical parameter ratio, b) is for the case of a bifurcating parameter ratio.

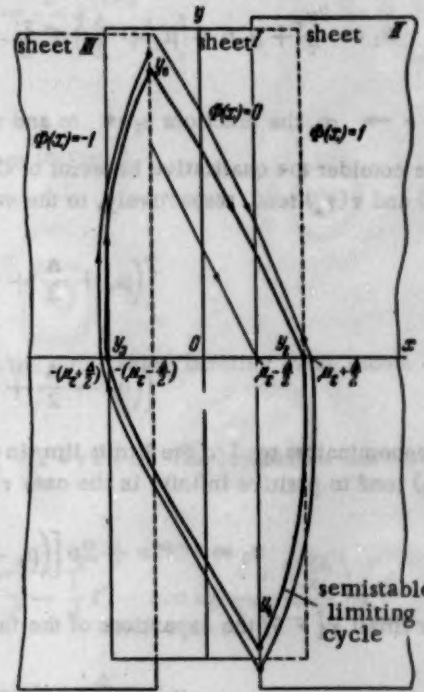

Fig. 8

Then, equating their derivatives, we find that the contact of the curves $u_1 = u_1(v)$ and $u_2 = u_2(v)$ occurs only on the line $v = 0$ (i.e., on $y_2 = 0$, since $v = \frac{y_2}{w}$). The boundary in $(\gamma, \Delta, \mu_\varepsilon)$ space, exactly defined by the analytic expression for the critical parameter ratio, $u_1(\tau) = u_2(\tau_1)$, $v(\tau) = 0$, $v(\tau_1) = 0$, divides the region of stability from the region of auto-oscillation. However, the construction of this boundary in $(\gamma, \Delta, \mu_\varepsilon)$ space is rendered difficult because of the transcendence of the equations.

Figure 7, a gives the diagram of the point transformation in the case of a semistable limiting cycle.

Thus, the semistable cycle on the (x, y) phase space will pass through the boundary of the dead zone (Fig. 8).

For the case $\gamma > \frac{1}{4}$ the function corresponding to (A) is found from (8), and the function corresponding to (B) is found from (12). We investigate the behavior of the curves $u_1 = u_1(\tau)$, $v = v(\tau)$ and $u_2 = u_2(\tau_1)$, $v = v(\tau_1)$, defined by these expressions. For this, we consider the qualitative behavior of Curve (8). The expressions for the derivatives may be written in the form

$$\frac{du_1}{d\tau} = -\frac{v}{\sin\tau} e^{p\tau}, \quad \frac{dv}{d\tau} = -\frac{u_1}{\sin\tau} e^{-p\tau}. \tag{17}$$

and, consequently,

$$\frac{du_1}{dv} = \frac{v}{u_1} e^{2p\tau},$$

$$\frac{d^2u_1}{dv^2} = \frac{2(p^2+1)e^{3p\tau}}{u_1^3}\left[\left(\mu_\varepsilon^2 + \frac{\Delta^2}{4}\right)\operatorname{sh} p\tau + \left(\mu_\varepsilon^2 - \frac{\Delta^2}{4}\right)p\sin\tau + \mu_\varepsilon\Delta\operatorname{ch} p\tau\right]. \tag{18}$$

It follows from (18) that, in the first quadrant, the derivatives $\frac{du_1}{dv}$ and $\frac{d^2u_1}{dv^2}$ are always positive as τ varies from zero to infinity and, consequently, the curve $u_1 = u_1(v)$ remains concave as τ varies within those limits.

For small $\tau > 0$, the expansions of the functions $u_1 = u_1(\tau)$ and $v = v(\tau)$ have the form

$$u_1 = \frac{2\mu_\varepsilon}{\tau} + \mu_\varepsilon p + \left(\mu_\varepsilon + \frac{\Delta}{2}\right)p^2\frac{\tau}{2} + \ldots, \quad v = \frac{2\mu_\varepsilon}{\tau} - \mu_\varepsilon p + \left(\mu_\varepsilon - \frac{\Delta}{2}\right)p^2\frac{\tau}{2} + \ldots,$$

and, for $\tau \to \infty$, the functions $u_1 \to \infty$ and $v \to \infty$. The asymptotic curve of (8) is the line $u_1 = v + 4p\mu_\varepsilon$.

We consider the qualitative behavior of Curve (12). It follows from (12) that, as $\tau_1 \to \pi$, the numerators of $u_2(\tau_1)$ and $v(\tau_1)$ tend, respectively, to the values

$$\left[\left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)e^{-p\pi} + \left(\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right)\right]$$

and

$$\left[\left(\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right)e^{p\pi} + \left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\right],$$

and the denominators tend to the limits $\lim \sin \tau_1 = \pm 0$ for $\tau_1 \to \pi \mp 0$. Therefore, the functions $u_2(\tau_1)$ and $v(\tau_1)$ tend to positive infinity in the case $\tau_1 \to \pi - 0$. The asymptotic curve of (12) is the line

$$u_2 = e^{-p\pi}v + 2p\left[\left(\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right) + e^{-p\pi}\left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\right].$$

For small $\tau_1 > 0$, the expansions of the functions $u_2(\tau_1)$ and $v(\tau_1)$ have the form

$$u_2 = \frac{\Delta}{\tau_1} - \Delta p + \left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)p^2\frac{\tau_1}{2} + \ldots,$$
$$v = -\frac{\Delta}{\tau_1} - \Delta p + \left(\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right)p^2\frac{\tau_1}{2} + \ldots.$$

The expressions for the first and second derivatives have the form

$$\frac{du_2}{dv} = \frac{v}{u_2}e^{-2p\tau_1},$$

$$\frac{d^2u_2}{dv^2} = \frac{2(p^2+1)e^{-3p\tau_1}}{u_2^3}\left\{-\left[\left(\mu_\varepsilon + \frac{1}{\gamma}\right)^2 + \frac{\Delta^2}{4}\right]\sin p\tau_1 + \right. \tag{19}$$
$$\left. + \left[\left(\mu_\varepsilon + \frac{1}{\gamma}\right)^2 - \frac{\Delta^2}{4}\right] p \sin \tau_1 + \left(\mu_\varepsilon + \frac{1}{\gamma}\right)\Delta \operatorname{ch} p\tau_1\right\}.$$

It follows from (19) that, in the first quadrant, the derivative $\frac{du_2}{dv}$ is always positive as τ_1 varies from zero to infinity. The derivative $\frac{d^2u_2}{dv^2}$ may be either negative or positive, depending on the value of τ_1, that is, Curve (12) has at least one point of inflection.*

It thus follows from an analysis of the function $u_2 = u_2(v)$ that the curve corresponding to (B) is the branch of Curve (12) for $0 < \tau_1 < \pi$.

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* Indeed, for small τ_1 the derivative $\frac{d^2u_2}{dv^2}$ is positive [Curve (12) is concave], since the expression in curly brackets in Equation (19) is positive, having the form $\left\{\left(\mu_\varepsilon + \frac{1}{\gamma}\right)\Delta - \frac{\Delta^2}{2}p\tau_1\right\}$, for small τ_1.

The qualitative behavior, as investigated above, of the curves $u_1 = u_1(v)$ and $u_2 = u_2(v)$, in the case $\gamma > \frac{1}{4}$, shows that contact of these curves, satisfying, respectively, the equations

$$\frac{du_1}{dv} = \frac{v}{u_1} e^{2p\tau} \quad \text{and} \quad \frac{du_2}{dv} = \frac{v}{u_2} e^{-2p\tau_1},$$

also occurs just on the line $v = 0$ (i.e., on $y_2 = 0$, since $v = \frac{y_2}{w}$).

The semistable limiting cycle for the case $\gamma > \frac{1}{4}$ is shown in Fig. 9.

5. Critical Ratio of the System Parameters

A. Case of a Large Load, $\gamma > \frac{1}{4}$

We shall analyze the motion of the representative point on the semistable limiting cycle shown in Fig. 9.

1. Equations of the integral curves on sheet I

The equation of motion of System (3) for sheet I will be $\ddot{x} + \dot{x} + \gamma x = 0$. The solution of this equation has the form

$$x = x_0 \sqrt{\frac{\gamma + \frac{y_0}{x_0} + \frac{y_0^2}{x_0^2}}{\gamma - \frac{1}{4}}}\, e^{-\frac{1}{2}t^*} \cos\left(\sqrt{\gamma - \frac{1}{4}}\, t^* - \operatorname{arc\,tg} \frac{1 + 2\frac{y_0}{x_0}}{\sqrt{4\gamma - 1}}\right), \tag{20}$$

where x_0 and y_0 are the initial coordinates of the representative point on sheet I.

We find the time t^*_{1k} during which the representative point goes to the point $x = x_0$, $y = y_2 = 0$.

Using (20), we find

$$y_2 = \left(\frac{dx}{dt}\right)_{t^* = t^*_{1k}} = x_0 \sqrt{\frac{\left(\gamma + \frac{y_0}{x_0} + \frac{y_0^2}{x_0^2}\right)}{\gamma - \frac{1}{4}}}\, e^{-\frac{1}{2}t^*_{1k}} \times$$

$$\times \cos\left(\sqrt{\gamma - \frac{1}{4}}\, t^*_{1k} - \operatorname{arc\,tg} \frac{1 + 2\frac{y_0}{x_0}}{\sqrt{4\gamma - 1}} - \operatorname{arc\,tg} \sqrt{4\gamma - 1}\right) = 0.$$

Here

$$\sqrt{\frac{\left(\gamma + \frac{y_0}{x_0} + \frac{y_0^2}{x_0^2}\right)\gamma}{\gamma - \frac{1}{4}}} \neq 0$$

and, so that the equality $y_2 = 0$ may hold, we must set*

$$\sqrt{\gamma - \frac{1}{4}}\, t^*_{1k} - \operatorname{arc\,tg} \frac{1 + 2\frac{y_0}{x_0}}{\sqrt{4\gamma - 1}} - \operatorname{arc\,tg} \sqrt{4\gamma - 1} = \frac{\pi}{2}.$$

* The equality $y_2 = 0$ also holds when $t^*_{1k} = \infty$. However, this case is of no interest.

Then, the time for the representative point to move to the point (x_2, y_2) is found from the expression

$$t^{\bullet}_{1k}=\frac{\pi+2\operatorname{arc\,tg}\dfrac{-1-2\gamma\dfrac{x_0}{y_0}}{\sqrt{4\gamma-1}}}{\sqrt{4\gamma-1}}. \tag{21}$$

Substituting (21) in (20), and bearing in mind that

$$\cos\left(\sqrt{\gamma-\frac{1}{4}}\,t^{\bullet}_{1k}-\operatorname{arc\,tg}\frac{1+2\dfrac{y_0}{x_0}}{\sqrt{4\gamma-1}}\right)=$$

$$=\cos\left(\frac{\pi}{2}+\operatorname{arc\,tg}\sqrt{4\gamma-1}\right)=-\sqrt{\frac{4\gamma-1}{4\gamma}},$$

we obtain, after some simple transformations, the expression for x_2

$$x_2=-x_0\sqrt{\frac{1}{\gamma}\frac{y_0^2}{x_0^2}+\frac{y_0}{x_0}+1}\exp\left[\frac{-1}{\sqrt{4\gamma-1}}\left(\frac{\pi}{2}+\operatorname{arc\,tg}\frac{-1-2\dfrac{x_0}{y_0}}{\sqrt{4\gamma-1}}\right)\right]. \tag{22}$$

2. Equations of the integral curves on sheet III

The equation of motion for System (4) for sheet III will be $\ddot{x}+\dot{x}+\gamma x=1$. The solution of this equation has the form

$$x=\left(x_3-\frac{1}{\gamma}\right)\sqrt{\frac{4\gamma}{4\gamma-1}}\,e^{-\frac{1}{2}t^{\bullet}}\cos\left(\sqrt{\gamma-\frac{1}{4}}\,t^{\bullet}-\operatorname{arc\,tg}\frac{1}{\sqrt{4\gamma-1}}\right), \tag{23}$$

where x_3 is the initial coordinate of the representative point on sheet III ($x = x_3$, $y = y_3 = 0$).

The final value of the coordinate $\underline{x}$ on sheet III is found from the expression

$$x_0=\left(x_3-\frac{1}{\gamma}\right)\sqrt{\frac{4\gamma}{4\gamma-1}}\,e^{-\frac{1}{2}t^{\bullet}_{3\text{k}}}\cos\left(\sqrt{\gamma-\frac{1}{4}}\,t^{\bullet}_{3\text{k}}-\operatorname{arc\,tg}\frac{1}{\sqrt{4\gamma-1}}\right), \tag{24}$$

where $t^{\bullet}_{3k}$ is the time for the representative point to get from the point (x_3, y_3) to the point (x_0, y_0).

If we replace $\cos\beta$ in Equation (24) by $\dfrac{(e^{j\beta}+e^{-j\beta})}{2}$ where $\beta=\sqrt{\gamma-\dfrac{1}{4}}\,t^{\bullet}_{3k}-\arctan\dfrac{1}{\sqrt{4\gamma-1}}$, then expand the exponential functions in powers of $t^{\bullet}_{3k}$, limiting the series to three terms each, we obtain

$$t^{\bullet 2}_{3\text{k}}-2\left(1-\sqrt{4\gamma-1}\operatorname{arc\,tg}\frac{1}{\sqrt{4\gamma-1}}\right)\frac{1}{2\gamma-1}\,t^{\bullet}_{3\text{k}}-$$

$$-\left[4-2\left(-\operatorname{arc\,tg}\frac{1}{\sqrt{4\gamma-1}}\right)^2-\frac{4\left(x_0-\dfrac{1}{\gamma}\right)}{x_3-\dfrac{1}{\gamma}}\sqrt{\frac{4\gamma-1}{4\gamma}}\right]\frac{1}{2\gamma-1}=0.$$

After substitution of the values for x_0 and x_3, we get the following form for the expression thus found for $t^{\bullet}_{3k}$:

$$t^{\bullet}_{3\text{k}}=\left(1-\sqrt{4\gamma-1}\operatorname{arc\,tg}\frac{1}{\sqrt{4\gamma-1}}\right)\frac{1}{2\gamma-1}+$$

$$+\left\{\left(1-\sqrt{4\gamma-1}\operatorname{arc\,tg}\frac{1}{\sqrt{4\gamma-1}}\right)^2\frac{1}{(2\gamma-1)^2}+\left[4-2\left(-\operatorname{arc\,tg}\frac{1}{\sqrt{4\gamma-1}}\right)^2-\right.\right.$$

$$\left.\left.-\frac{4\left(\mu_\text{s}-\dfrac{\Delta}{2}+\dfrac{1}{\gamma}\right)}{\mu_\text{s}+\dfrac{\Delta}{2}+\dfrac{1}{\gamma}}\sqrt{\frac{4\gamma-1}{4\gamma}}\right]\frac{1}{2\gamma-1}\right\}^{1/2}. \tag{25}$$

We get the expression for the coordinate y_0 by differentiating Equation (23) with respect to the time t^*, and substituting the values for x_3 and $t^* = t^*_{3k}$ in the expression for the derivative. Taking into account that

$$\cos\left(\sqrt{\gamma-\frac{1}{4}}\; t^{*}_{\text{зн}} - \operatorname{arc\,tg}\frac{1}{\sqrt{4\gamma-1}} - \operatorname{arc\,tg}\sqrt{4\gamma-1}\right) = \sin\left(t^{*}_{\text{зн}}\sqrt{\gamma-\frac{1}{4}}\right),$$

we can write the desired expression in the form

$$y_0 = -\left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\frac{2\gamma}{\sqrt{4\gamma-1}}\exp\left(-\frac{t^{*}_{\text{зн}}}{2}\right)\sin\left(t^{*}_{\text{зн}}\sqrt{\gamma-\frac{1}{4}}\right). \tag{26}$$

3. Critical ratio of the parameters

The semistable limiting cycle shown in Fig. 9 goes through sheets III and I of the phase surface through the points with the coordinates

$$x_3 = -\mu_\varepsilon - \frac{\Delta}{2},\quad y_3 = 0,\quad x_0 = -\mu_\varepsilon + \frac{\Delta}{2},$$

y_0, as determined from Equation (26),

$$x_2 = \mu_\varepsilon + \frac{\Delta}{2},\quad y_2 = 0. \tag{27}$$

Substituting (26) and (27) in (22), we get the critical ratio of the parameters

$$\begin{aligned}\left(\mu_\varepsilon + \frac{\Delta}{2}\right)^2 = \Bigg[&\left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)^2\frac{4\gamma}{4\gamma-1}e^{-t^{*}_{\text{зн}}}\sin^2\left(t^{*}_{\text{зн}}\sqrt{\gamma-\frac{1}{4}}\right) + \\ &+\frac{1}{\gamma}\left(\mu_\varepsilon - \frac{\Delta}{2}\right)\left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\sqrt{\frac{4\gamma^2}{4\gamma-1}}\,e^{-\frac{t^{*}_{\text{зн}}}{2}}\sin\left(t^{*}_{\text{зн}}\sqrt{\gamma-\frac{1}{4}}\right) + \\ &+\left(-\mu_\varepsilon + \frac{\Delta}{2}\right)^2\Bigg]\exp\left\{-\frac{1}{\sqrt{4\gamma-1}}\left[\pi + 2\operatorname{arc\,tg}\times\right.\right. \\ &\left.\left.\times\left(-\frac{1}{\sqrt{4\gamma-1}} + \frac{\left(-\mu_\varepsilon + \frac{\Delta}{2}\right)e^{\frac{t^{*}_{\text{зн}}}{2}}}{\left(\mu_\varepsilon + \frac{\Delta}{2} + \frac{1}{\gamma}\right)\sin\left(t^{*}_{\text{зн}}\sqrt{\gamma-\frac{1}{4}}\right)}\right)\right]\right\}.\end{aligned} \tag{28}$$

The expression thus obtained may be simplified by replacing in it the functions $e^{-t^*_{3k}}$, $e^{-\frac{t^*_{3k}}{2}}$, and $e^{\frac{t^*_{3k}}{2}}$ by their values determined from Expression (24), in which x_0 and x_3 are replaced by their values determined from (27). Then, finally, we obtain the critical parameter ratio for $(\gamma,\ \mu_\varepsilon,\ \Delta)$, in the case $\gamma > \frac{1}{4}$, in the following form:

$$\left(\mu_\varepsilon + \frac{\Delta}{2}\right)^2\exp\left\{A_1\pi + 2A_1\operatorname{arctg}\left[-A_1 - \frac{\mu_\varepsilon - \frac{\Delta}{2}}{\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma}}(\operatorname{ctg}A_2 + A_1)\right]\right\} -$$

$$-\left(-\mu_\varepsilon + \frac{\Delta}{2}\right)^2 = \frac{A_3 + A_4\operatorname{ctg}A_2}{A_5(\operatorname{ctg}A_2 + A_1)^2},$$

where

$$A_1 = \frac{1}{\sqrt{4\gamma - 1}}, \quad A_3 = 4\gamma\left(\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right)^2 + 2\left(\mu_\varepsilon - \frac{\Delta}{2}\right)\left(\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right),$$

$$A_2 = t^*_{3k}\sqrt{\gamma - \frac{1}{4}}, \quad A_4 = 2\left(\mu_\varepsilon - \frac{\Delta}{2}\right)\left(\mu_\varepsilon - \frac{\Delta}{2} + \frac{1}{\gamma}\right)\sqrt{4\gamma - 1},$$

$$A_5 = 4\gamma - 1,$$

and the parameter t^*_{3k} is defined by Expression (25).

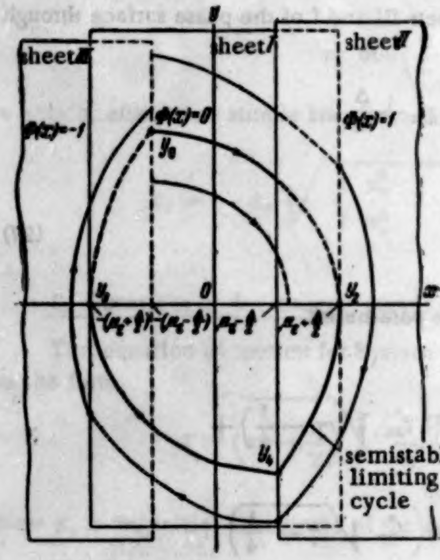

Fig. 9

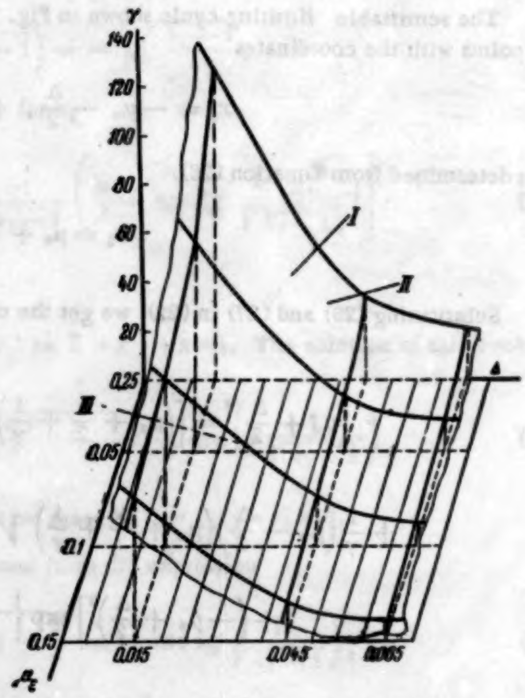

Fig. 10. (γ, μ_ε, Δ) parameter space for the case $\gamma > \frac{1}{4}$: I is the surface $F(\gamma, \mu_\varepsilon, \Delta)$, II is the region of stability, III is the region of auto-oscillation.

Figure 10 gives, for the case $\gamma > \frac{1}{4}$, the decomposition of (γ, μ_ε, Δ) parameter space into regions corresponding to auto-oscillations of the servomechanism, and to their absence.

B. Case of a Small Load, $\gamma < \frac{1}{4}$

We shall now analyze the motion of the representative point on the semistable limiting cycle shown in Fig. 8.

1. Equations of the integral curves on sheet I

The expression for the coordinate x on sheet I as a function of time t^*, and of the initial conditions x_0 and y_0, has the form

$$x = \frac{y_0 + q_2 x_0}{q_2 - q_1} e^{-q_1 t^*} + \frac{y_0 + q_1 x_0}{q_1 - q_2} e^{-q_2 t^*}, \tag{29}$$

where

$$-q_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} - \gamma},$$

$$-q_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} - \gamma}.$$

The time t^*_{1d}, taken by the representative point to pass from the point (x_0, y_0) to the point $(x_1, y_1 = 0)$ is found by differentiating Equation (29) with respect to the time t^*, and expanding the exponential functions in series limited to the first three terms of the expansions.

As the result, we get

$$\overset{*}{t}{}_{1d}^{2} - \frac{2\left(2y_0 + \frac{1}{2}x_0\right)}{(1-\gamma)\,y_0 + 2\gamma x_0}\,\overset{*}{t}_{1d} - \frac{4y_0}{(1-\gamma)\,\gamma_0 + 2\gamma x_0} = 0,$$

from which we find the desired time

$$\overset{*}{t}_{1d} = \frac{2y_0 + \frac{1}{2}x_0}{(1-\gamma)\,y_0 + 2\gamma x_0}\left\{\left[\frac{2y_0 + \frac{1}{2}x_0}{(1-\gamma)y_0 + 2\gamma x_0}\right]^2 + \frac{4y_0}{(1-\gamma)\,y_0 + 2\gamma x_0}\right\}^{1/2}. \tag{30}$$

Substituting the value of t^*_{1d} from (30) into (29), we obtain the expression for the coordinate x_2 in terms of the coordinates x_0 and y_0:

$$x_2 = \left[y_0 + x_0\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \gamma}\right)\right]\frac{1}{\sqrt{1-4\gamma}}\exp\left[\left(-\frac{1}{2} + \sqrt{\frac{1}{4} - \gamma}\right)\overset{*}{t}_{1d}\right] -$$
$$- \left[y_0 + x_0\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \gamma}\right)\right]\frac{1}{\sqrt{1-4\gamma}}\exp\left[\left(-\frac{1}{2} - \sqrt{\frac{1}{4} - \gamma}\right)\overset{*}{t}_{1d}\right]. \tag{31}$$

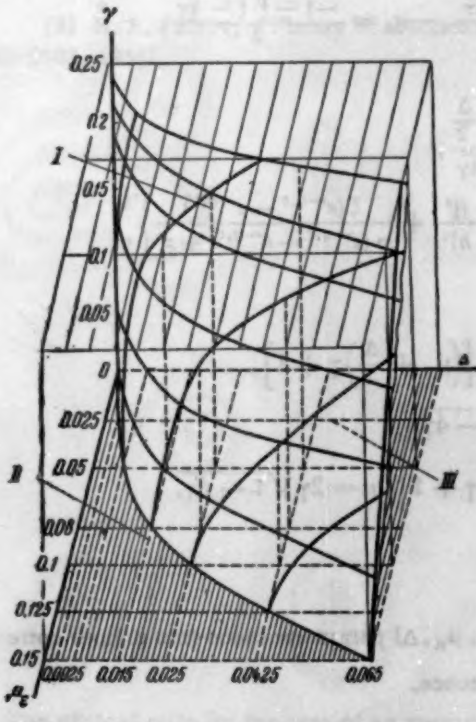

Fig. 11. (γ, μ_ϵ, Δ) parameter space for the case $\gamma < \frac{1}{4}$: I is the surface F (γ, μ_ϵ, Δ), II is the region of stability, III is the region of auto-oscillation.

2. Equations of the integral curves on sheet III

In the case $\gamma < \frac{1}{4}$, the expression for the coordinate x_0 in terms of the initial conditions x_3, $y_3 = 0$, and of the time t^*_{3d} for the representative point to move from the point (x_3, y_3) to the point (x_0, y_0), has the form

$$x_0 = \frac{\left(x_3 - \frac{1}{\gamma}\right)q_2}{q_2 - q_1}\,e^{-q_1\overset{*}{t}_{3d}} -$$
$$- \frac{\left(x_3 - \frac{1}{\gamma}\right)q_1}{q_2 - q_1}\,e^{-q_2\overset{*}{t}_{3d}} + \frac{1}{\gamma}, \tag{32}$$

where q_1 and q_2 are the same as in (29).

Substituting in (32) the values of x_0 and x_3 from (27), and of q_1 and q_2 from (29), and also expanding the exponential functions in series, keeping only the first three terms of the expansions, we get

$$\overset{*}{t}_{3d} = \sqrt{\frac{2\Delta}{\left(\mu_\epsilon + \frac{\Delta}{2}\right)\gamma + 1}}. \tag{33}$$

By differentiating (20) with respect to time t^*, and substituting the value $t^* = t^*_{3d}$, we obtain the value for y_0

$$y_0 = \frac{\left(\mu_\varepsilon + \frac{\Delta}{2}\right)\gamma + 1}{\sqrt{1-4\gamma}} \left\{ \exp\left[\left(-\frac{1}{2} + \sqrt{\frac{1}{4} - \gamma}\right) \sqrt{\frac{2\Delta}{\left(\mu_\varepsilon + \frac{\Delta}{2}\right)\gamma + 1}}\right] - \right.$$

$$\left. - \exp\left[\left(-\frac{1}{2} - \sqrt{\frac{1}{4} - \gamma}\right) \sqrt{\frac{2\Delta}{\left(\mu_\varepsilon + \frac{\Delta}{2}\right)\gamma + 1}}\right] \right\}. \tag{34}$$

3. Critical ratio of the parameters

By analogy with the case $\gamma > \frac{1}{4}$, we obtain the critical parameter ratio for $(\gamma, \mu_\varepsilon, \Delta)$ in the case $\gamma < \frac{1}{4}$ by substituting in (31) the value of y_0 from (34) and the values of x_0 and x_2 from (27). As the result, we obtain

$$\mu_\varepsilon + \frac{\Delta}{2} = [\delta(e^{-q_1 c} - e^{-q_2 c}) + dq_2]\, e^{-q_1 s} - [\delta(e^{-q_1 c} - e^{-q_2 c}) + dq_1]\, e^{-q_2 s}, \tag{35}$$

where

$$\delta = \frac{\left(\mu_\varepsilon + \frac{\Delta}{2}\right)\gamma + 1}{1 - 4\gamma}, \quad -q_1 = \frac{-1 + \sqrt{1-4\gamma}}{2}, \quad -q_2 = \frac{-1 - \sqrt{1-4\gamma}}{2},$$

$$c = t^*_{3d}, \quad d = \frac{-\mu_\varepsilon + \frac{\Delta}{2}}{\sqrt{1-4\gamma}},$$

$$s = \frac{k(e^{-q_1 c} - e^{-q_2 c}) + l}{m(e^{-q_1 c} - e^{-q_2 c}) + n} + \left\{ \frac{[k(e^{-q_1 c} - e^{-q_2 c}) + l]^2}{[m(e^{-q_1 c} - e^{-q_2 c}) + n]^2} + \frac{\xi(e^{-q_1 c} - e^{-q_2 c})}{m(e^{-q_1 c} - e^{-q_2 c}) + \rho} \right\}^{1/2},$$

taking

$$k = 2\left[\left(\mu_\varepsilon + \frac{\Delta}{2}\right)\gamma + 1\right], \quad m = (1-\gamma)\left[\left(\mu_\varepsilon + \frac{\Delta}{2}\right)\gamma + 1\right],$$

$$l = \frac{1}{2}\left(-\mu_\varepsilon + \frac{\Delta}{2}\right)\sqrt{1-4\gamma},$$

$$n = 2\gamma\left(-\mu_\varepsilon + \frac{\Delta}{2}\right)\sqrt{1-4\gamma}, \quad \xi = 4\left[\left(\mu_\varepsilon + \frac{\Delta}{2}\right)\gamma + 1\right], \quad \rho = 2\gamma\sqrt{1-4\gamma},$$

and the parameter t^*_{3d} is defined by Expression (33).

Figure 11 gives, for the case $\gamma < \frac{1}{4}$, the decomposition of $(\gamma, \mu_\varepsilon, \Delta)$ parameter space into regions corresponding to auto-oscillations of the servomechanism, and to their absence.

SUMMARY

1. In relay-type servomechanisms, the motions of which are described by complete second-order differential equations where the right members are in the form of relay functions whose characteristics contain loops and dead

zones, there exist two different critical parameter ratios:* one for large loads, (28), and one for small loads, (35).

2. It follows from the decomposition of $(\gamma, \Delta, \mu_\varepsilon)$** parameter space into regions of qualitatively different system behavior (Figs. 10 and 11) that an increase in the load, varying proportionally to the motion, aids the appearance of auto-oscillation in the system.

The present work was carried out under the supervision of B. N. Petrov and V. V. Petrov.

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* The critical ratio for the case of an average load, $\gamma = \frac{1}{4}$ (corresponding to equal roots), is of no interest, since, in practice, the characteristic equation of the system has different, unequal roots.

** Generally, $\left|\mu_\varepsilon \pm \frac{\Delta}{2}\right| < \frac{1}{\gamma}$.

STABILITY OF PERIODIC CONDITIONS IN AUTOMATIC CONTROL SYSTEMS FOUND APPROXIMATELY ON THE BASIS OF A FILTER HYPOTHESIS

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An approximate method of analyzing the stability of periodic conditions in nonlinear automatic control systems is outlined.

A brief comparison between the above method and that based on the autoresonance theory is given.

1. Introduction

When investigating nonlinear automatic control systems it is often expedient for determining oscillatory conditions to use a harmonic balance method [1], for instance in the form of Gol'dfarb's graphic representation [2].

The determination of oscillatory conditions consists in the first place in finding periodic solutions to a system of differential equations which represent the process, and in the second place in separating the stable periodic solutions from the unstable ones. The necessity of solving both parts of the problem in finding oscillatory conditions, applicable to systems of automatic control, is connected with the recently discovered [3 and 4] contradiction between the basic assumptions on which the harmonic balance methods rest and the dynamic properties of automatic control systems.

The harmonic balance method has been proposed and is valid for equations which contain the small parameter μ, so that when $\mu = 0$ the equation produces a harmonic periodic solution. In such a case, when μ is sufficiently small, the periodic solution, if it exists at all, differs in its form but little from a harmonic one, a circumstance which permits one to find a periodic solution in the form of a harmonic function. In addition the setting-up process of a periodic solution, when disturbed but little, differs from the periodic solution itself only by slow changes in the amplitude and phase with respect to time. The latter condition validates the stability investigation methods used, for instance, in Gol'dfarb's construction [2].

Equations representing the setting-up process do not, as a rule, contain the small parameter μ. In order to justify the application of Gol'dfarb's construction, other considerations are taken into account: it is assumed that the oscillations only contain frequencies whose harmonics lie outside the passband of the linear part of the system, this part being considered as a low-pass filter. These considerations suffice to justify the finding of periodic solutions in the form of harmonic functions. Hence, they justify the application of Gol'dfarb's construction for determining periodic solution in a certain frequency band.*

These considerations, however, do not provide reasons to believe that in the transient state the amplitude and phase of deviations from the steady-state periodic movement change slowly; hence, it is not valid to use the

* In the simplest case of an odd nonlinear characteristic this frequency band is equal to $\frac{\omega_{cp}}{3} < \omega < \omega_{cp}$, where ω_{cp} is the linear filter cutoff frequency.

harmonic balance method (in particular Gol'dfarb's construction) for investigating the stability of periodic conditions in such cases.

Thus arises the problem raised in [6]: assuming that the system under investigation contains a linear filter, to find the conditions of stability for periodic solutions, obtained on the basis of these assumptions.

An approximate solution of the problem is proposed below.

2. Equation for Small Deviations and Reduction of the Problem to a Characteristic Equation of an Infinite Degree

Let us examine a system of automatic control, consisting of an arbitrary linear part and a nonlinear feedback. The equations of the process will assume the form*

$$D(p)X = -K(p)Y,\ Y = F(X), \tag{1}$$

where $p = \frac{d}{dt}$, D and K are polynomials with constant coefficients and the power of K is lower than that of D.

It is assumed that $\frac{K(j\omega)}{D(j\omega)} = 0$ when $\omega > \omega_{cp}$, i.e., that ω_{cp} is the boundary of the linear filter passband. Function F(X) is assumed to be odd, but this limitation has been introduced to make the problem more concrete and it can be easily omitted.

For determining stability, let us introduce increments. Let

$$X = \overline{X(t)} + x. \tag{2}$$

Inserting (2) in (1), we find

$$D(p)\overline{X(t)} + D(p)x = -K(p)F[\overline{X(t)} + x].$$

By developing the nonlinear function into a power series in $\underline{x}$, neglecting the nonlinear terms for sufficiently small values of $\underline{x}$ and considering that X(t) satisfies Equation (1), we obtain

$$D(p)x + K(p)A_x(t)x = 0, \tag{3}$$

where

$$A_x(t) = \left[\frac{dF(x)}{dx}\right]_{X=\overline{X_s(t)}}$$

is the given periodic function of time (with period τ).

Let us represent periodic function (3) by its development into a Fourier series **

$$A_x(t) = A_x + A_x m_{1x}\cos(\Omega t + \alpha_{1x}) + A_x m_{2x}\cos(2\Omega t + \alpha_{2x}) + \ldots \tag{4}$$

For the investigated periodic condition X(t), determined by Equation (1), to be stable it is necessary and sufficient for the solution of the linear differential equation (3) with periodic coefficients to be stable.

Let us find a general solution of Equation (3) in the following form:

$$x(t) = \mathrm{Re}\sum_{k=-\infty}^{\infty} X_k e^{[\lambda t + j(k\Omega t + \varphi_k)]}, \tag{5}$$

where λ are proper numbers to be determined and X_k are the required amplitudes.

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* A more general case Y = F(X, X', X") can also be examined by means of methods similar to those described below.

** It should be noted that in (4) $\alpha_{kx} = k\alpha_{1x}$; hence, it is possible to get rid of quantity α_{kx} by suitably choosing the origin of the time basis.

By inserting (4) and (5) into (3) and taking, for simplicity, $m_{1x} = m_1$, after simple transformations, we shall obtain

$$A_0K(p)\Big[\sum_{k=-\infty}^{\infty} X_k e^{[\lambda t+j(k\Omega t+\varphi_k)]} + \frac{m_1e^{j\alpha_1}}{2}\sum_{k=-\infty}^{\infty} X_k e^{\{\lambda t+j[(k+1)\Omega t+\varphi_k]\}} +$$

$$+\frac{m_1e^{-j\alpha_1}}{2}\sum_{k=-\infty}^{\infty} X_k e^{\{\lambda t+j[(k-1)\Omega t+\varphi_k]\}} + \ldots + \frac{m_re^{j\alpha_r}}{2}\sum_{k=-\infty}^{\infty} X_k e^{\{\lambda t+j[(k+r)\Omega t+\varphi_k]\}} +$$

$$+\frac{m_re^{-j\alpha_r}}{2}\sum_{k=-\infty}^{\infty} X_k e^{\{\lambda t+j[(k-r)\Omega t+\varphi_k]\}} + \ldots\Big] + D(p)\sum_{k=-\infty}^{\infty} X_k e^{[\lambda t+j(k\Omega t+\varphi_k)]} = 0. \tag{6}$$

By changing in all the index sums $(k \pm r)$ the quantity $\underline{k}$ into $(k \mp r)$ and introducing notation $\dot{X}_k = X_k e^{j\varphi_k}$, we can rewrite (6) in the form

$$[A_0K(p)+D(p)]\sum_{k=-\infty}^{\infty}\dot{X}_k e^{(\lambda t+jk\Omega t)} + \frac{1}{2}A_0K(p)\,m_1e^{j\alpha_1}\sum_{k=-\infty}^{\infty}\dot{X}_{k-1}e^{(\lambda t+jk\Omega t)} +$$

$$+\frac{1}{2}A_0K(p)\,m_1e^{-j\alpha_1}\sum_{k=-\infty}^{\infty}\dot{X}_{k+1}e^{(\lambda t+jk\Omega t)} + \ldots + A_0K(p)\,m_re^{j\alpha_r}\times$$

$$\times\sum_{k=-\infty}^{\infty}\dot{X}_{k-r}e^{(\lambda t+jk\Omega t)} + \frac{1}{2}A_0K(p)\,m_re^{j\alpha_r}\sum_{k=-\infty}^{\infty}\dot{X}_{k+r}e^{(\lambda t+jk\Omega t)} + \ldots = 0. \tag{7}$$

If the complex amplitudes at similar frequencies are equated to zero, Equation (7) transforms into a system of an infinite number of equations, each with an infinite number of terms

$$\{[A_0K(p)+D(p)]\,e^{(\lambda+jk\Omega)t}\}\,\dot{X}_k + \Big[\frac{1}{2}A_0m_1e^{j\alpha_1}K(p)\,e^{(\lambda+jk\Omega)t}\Big]\dot{X}_{k-1} +$$

$$+\Big[\frac{1}{2}A_0m_1e^{-j\alpha_1}K(p)e^{(\lambda+jk\Omega)t}\Big]\dot{X}_{k+1} + \ldots + \Big[\frac{1}{2}A_0m_re^{j\alpha_r}K(p)\,e^{(\lambda+jk\Omega)t}\Big]\dot{X}_{k-r} +$$

$$+\Big[\frac{1}{2}A_0m_re^{-j\alpha_r}K(p)\,e^{(\lambda+jk\Omega)t}\Big]\dot{X}_{k+r} + \ldots = 0 \quad (k=-\infty,\ldots,-1,0,1,\ldots,\infty). \tag{8}$$

By equating to zero the determinant of this infinite system of equations we obtain a characteristic equation of an infinite power with respect to λ. Its roots are the required values of λ. For the investigated periodic condition to be stable it is necessary and sufficient that all these roots have a negative real part. Obviously, in this form the conditions of stability cannot be applied in practice. But we have not yet used the filter conditions, which can now be applied.

3. Filter Conditions and Limitation of the Power of the Characteristic Equation

If it is taken into account that the cutoff frequency is finite and the complex amplitudes of components whose frequencies exceed the cutoff, are equal to zero, the infinite system of equations can be converted into a finite one. In fact all complex amplitudes, corresponding to the value $\underline{k}$ which satisfies the condition

$$|\operatorname{Im}\lambda_i \pm k\Omega| \geqslant \omega_{cp}, \tag{9}$$

will be identically equal to zero.

Taking into consideration that the λ_i are unknown, it is possible to substitute condition (9), if the sign in front of $k\Omega$ is appropriately chosen, by the approximate condition

$$k_{\lim}\Omega \geqslant \omega_{cp}. \tag{10}$$

Hence, the limiting expression of $k = k_{lim}$, which corresponds to the complex amplitude equal to zero, will be determined as

$$k_{\text{lim}} \geqslant \frac{\omega_{\text{cp}}}{\Omega}. \tag{11}$$

If, for instance, $\omega_{cp} = 2\Omega$, we shall obtain a finite system of five equations instead of an infinite system (8).

If we denote the diagonal term coefficient (with a complex amplitude $\dot{X}_k$) by $a_{k,k}$ and the mutual coefficient at the crossing of the $\underline{k}$ line with the $(k \pm r)$ column by $a_{k,k\pm r}$, and take $\omega_{cp} = 2\,\Omega$, we shall obtain from (8) a finite system of equations which in a developed form will be written down as

$$\begin{aligned}
&a_{-2,-2}\dot{X}_{-2} + a_{-2,-1}\dot{X}_{-1} + a_{-2,0}\dot{X}_0 + a_{-2,1}\dot{X}_1 + a_{-2,2}\dot{X}_2 = 0,\\
&a_{-1,-2}\dot{X}_{-2} + a_{-1,-1}\dot{X}_{-1} + a_{-1,0}\dot{X}_0 + a_{-1,1}\dot{X}_1 + a_{-1,2}\dot{X}_2 = 0,\\
&a_{0,-2}\dot{X}_{-2} + a_{0,-1}\dot{X}_{-1} + a_{00}\dot{X}_0 + a_{0,1}\dot{X}_1 + a_{0,2}\dot{X}_2 = 0,\\
&a_{1,-2}\dot{X}_{-2} + a_{1,-1}X_{-1} + a_{1,0}\dot{X}_0 + a_{1,1}\dot{X}_1 + a_{1,2}\dot{X}_2 = 0,\\
&a_{2,-2}\dot{X}_{-2} + a_{2,-2}\dot{X}_{-1} + a_{2,0}\dot{X}_0 + a_{2,1}\dot{X}_1 + a_{2,2}\dot{X}_2 = 0.
\end{aligned} \tag{12}$$

According to previous statements the coefficients in (12) will be, respectively, equal to

$$\begin{aligned}
a_{k,k} &= [A_0 K(p) + D(p)]\, e^{(\lambda_i + jk\Omega)t},\\
a_{k,k\pm r} &= \frac{1}{2} A_0 m_r e^{\pm j\alpha_r} K(p)\, e^{(\lambda_i + jk\Omega)t}.
\end{aligned} \tag{13}$$

If the expression for the coefficient $\left|\frac{dF(X)}{dX}\right|_{X=X(t)}$, expanded into a Fourier series, is limited to a finite number of terms equal to ν, all the $a_{k,k\pm r}$ terms with $r > \nu$ will become equal to zero. Thus, with some of the terms becoming zero, Equation (12) will be simplified.

In equating to zero the determinant of the final system of equations with complex amplitudes we shall obtain a polynomial in λ with complex coefficients. The roots of this polynomial are equal to the required values of λ_i.

For the periodic state under investigation to be stable, it is necessary and sufficient that all Re $\lambda_i > 0$. When investigating whether all the λ_i satisfy this condition, it is possible to use both algebraic and frequency criteria of stability.

Taking into consideration that in this instance roots of a polynomial with complex coefficients are being investigated, it is possible to use Shur's criterion when applying algebraic criteria of stability.

When using frequency methods it is possible to apply, for instance, Mikhailov's criterion (more precisely, the principle of his reasoning, since in this case the complex roots of the characteristic equation are not conjugate).

4. Comparison with the Stability Conditions of the Harmonic Balance Method

In conclusion let us compare the conditions of stability derived on the basis of assuming the existence of a filtering action of the linear part of the system, with the known conditions proposed by Gol'dfarb [2] and based on the harmonic balance method.

As M. A. Aizerman and L. M. Smirnova [4] have shown, the conditions of stability used in the harmonic balance method can be obtained if it is assumed that the system is only slightly nonlinear and that it is therefore possible to find small deviations from the steady periodic state in the form of $x(t) = c(t) \sin[\omega t + \gamma(t)]$, where $c(t)$ and $\gamma(t)$ are slowly changing quantities, and that the following averaging over one period is permissible:

$$\begin{aligned}
f\,[(c_0 + \Delta c)\sin(\omega t - \gamma_0 - \Delta\gamma)] &\approx (c_0 + \Delta c)\, g(c_0 + \Delta c)\sin(\omega t - \gamma_0 - \Delta\gamma) +\\
&+ (c_0 + \Delta c)\, b(c_0 + \Delta c)\cos(\omega t - \gamma_0 - \Delta\gamma),
\end{aligned}$$

where Δc and $\Delta\gamma$ are small quantities.

Let us find on the basis of the filter hypothesis the approximate stability conditions for the case when the cutoff frequency satisfies condition $0 < \omega_{cp} \leq \Omega$.

In this case, according to (12) and (13),

$$\begin{vmatrix} [A_0 K(p) + D(p)]\, e^{(\lambda_1 - j\Omega)t}, & \frac{1}{2} A_0 m_1 e^{j\alpha_1} K(p)\, e^{(\lambda - j\Omega)t} \\ \frac{1}{2} A_0 m_1 e^{-j\alpha_1} K(p)\, e^{\lambda t}, & [A_0 K(p) + D(p)]\, e^{\lambda t} \end{vmatrix} = 0. \tag{14}$$

By solving the determinant we shall obtain a polynomial in λ. In evaluating the roots of this polynomial it becomes possible to answer the question of the periodic movement stability. If $\operatorname{Re}\lambda < 0$, the movement will be stable.

It is easily seen that these conditions of stability differ from those obtained by the autoresonance hypothesis.

Let us now examine the relation between the stability conditions obtained by the filter hypothesis and those obtained by the autoresonance method.

Let us make assumptions similar to those adopted in the autoresonance method, namely, let us assume that the system is only slightly nonlinear, i.e., that it is possible to neglect terms containing m^2 and that owing to the slow change in the amplitude it is possible to neglect terms containing powers of λ greater than one. Under these conditions determinant (14) will assume the form

$$A_0\left\{a_n + \left[\sum_{r=1}^{n} a_{n-r} r (j\Omega)^{r-1}\right]\lambda + b_m + \left[\sum_{s=1}^{m} b_{m-s}\, s\,(j\Omega)^{s-1}\right]\lambda\right\}\left[A_0(a_{n-1}\lambda + a_n) + (b_{m-1}\lambda + b_m)\right] - \frac{1}{4}A_0^2 m_1^2\left\{a_n + \left[\sum_{r=1}^{n} a_{n-r} r (j\Omega)^{r-1}\right]\lambda\right\} = 0. \tag{15}$$

By solving the determinant, neglecting terms containing m^2 or powers of λ greater than one, we shall see that the stability condition will then be reduced to the condition that the real part of the coefficient of λ must be positive. When expanded this condition will be written down in the form

$$\operatorname{Re}\left\{A_0 a_n\left[A_0 a_{n-1} + b_{m-1} + \sum_{r=1}^{r} a_{n-r} r (j\Omega)^{r-1} + \sum_{s=1}^{m} b_{m-s}\, s\,(j\Omega)^{s-1}\right] + A_0 b_m\left[A_0 a_{n-1} + b_{m-1} + \sum_{r=1}^{r} a_{n-r} r (j\Omega)^{r-1} + \sum_{s=1}^{m} b_{m-s}\, s\,(j\Omega)^{s-1}\right]\right\} > 0. \tag{16}$$

Above condition, even with an assumed narrow passband of the linear part of the system [only the harmonic of frequency ($\operatorname{Im}\lambda_1 - j\Omega$) was taken into account], differs from the stability condition obtained by the autoresonance method.

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A METHOD OF ANALYZING AND CALCULATING TRANSIENT PROCESSES IN AUTOMATIC CONTROL OF GENERATOR EXCITATION BY MEANS OF A MAGNETIC AMPLIFIER

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The author proposes a method of calculating the equivalent time constant of a unit consisting of the operating winding of a magnetic amplifier and a single-cycle bridge-type rectifier with an inductive load, simulating a generator excitation winding.

1. Peculiarities of Transient and Stable-State Processes in Inductive Circuits With Rectifiers

Transient processes in nonlinear circuits, containing semiconductor rectifiers are, as a rule, represented by periodic piece-wise-continuous functions. For the mathematical representation of such processes it is more common at present to use the method of successive solutions of differential equations, mostly transcendental or nonlinear. The solutions are obtained for a large number of discrete parts of the process, with a subsequent matching of end and starting conditions of neighboring parts.

The existing methods of investigating stable-state processes in circuits with semiconductor rectifiers, including the method used in [1], are based on the application of the harmonic balance method. Such treatment is possible because the stable-state processes in these circuits are represented by periodic piece-wise-continuous functions which can be expanded into a harmonic series. The existing methods of investigating stable-state processes in above circuits do not follow from the consideration of transient processes. These circumstances prevent the application of such methods for investigating the dynamic operation of semiconductor rectifiers, in particular single-phase bridge-type rectifiers in circuits for the automatic control of electric generator excitation.

On the other hand, the existing methods of investigating transients in magnetic amplifiers, as applied for instance in the paper of L. A. Bessonov and M. A. Rozenblat, include the operation of magnetic amplifiers with a load on the ac side only, i.e., do not take into account the special transient conditions of operating jointly magnetic amplifiers and rectifiers.

Therefore, the results of investigation of transients in magnetic amplifiers with an ac load cannot be directly applied to investigating the dynamics of the automatic control of electrical machines in which the rectifier, feeding the highly inductive excitation winding, is an indispensable component.

All this shows how essential it is to develop methods of analyzing and calculating transient processes in inductive circuits with rectifiers, as applied to electrical machine automatic control systems which contain magnetic amplifiers.

Article [2] only deals with the operation of a magnetic amplifier without a feedback to an inductive load through a bridge-type rectifier. Such amplifiers are not normally used in automatic control systems.

In addition, in the above article it is assumed that the core of the magnetic amplifier has an ideal rectangular magnetization characteristic, which does not correspond to the actual condition in the circuit. In electrical machine

automatic control systems the core of the high-powered output amplifier, which is directly connected to the excitation winding of the machine, is made of electrical steel.

The publication of certain results of paper [3] in this article serves as an attempt to fill the gap in the technical literature on this subject.

The main peculiarity of inductive circuits containing rectifiers consists in sharp changes in their characteristics, due to switching, i.e., in rapid, practically instantaneous changes in circuit parameters. These changes are often accompanied by a change in the path taken by the current. For examining such processes it was found expedient to use as the basic quantity in calculations the definite integral of tension with respect to time $p_u = \int_{t_1}^{t} u(\tau)d\tau$, known as the tension pulse [4]. The tension pulse has the dimensions of magnetic flux (in sec) and can be measured with a fluxmeter.

An inductive linear or nonlinear circuit, with tension $u(t)$ impressed on it, can be represented by an equation of electrical balance

$$\frac{d\Psi}{dt} + Ri = u(t).$$

By integrating this equation term by term in the interval of $u(t)$ operation $t_1 \leq t \leq t_2$, we shall obtain an equation for pulses

$$\Delta\Psi + \int_{t_1}^{t_2} Ridt = p_u.$$

In the pulse equation, as applied to the control of electrical machines excitation, $\Delta\Psi$ represents the useful part of the pulse, which forms the flux, and $\int_{t_1}^{t_2} Ridt$ the tension pulse losses.

The transient process in inductive circuits with rectifiers is characterized by the effect of a large number of tension pulses, which are relatively short compared with the circuit time constant. In order to represent this process it is necessary to find the relation between the pulse equation terms in neighboring time intervals.

The basis of the investigation method consists in a precise solution of the transient process problem which arises when an ideal single-phase bridge-type rectifier circuit with two inductors L_1 and L_2 is connected to an alternating tension, which does not contain a direct component and is, in the general case, nonsinusoidal [3]. Inductor L_1 is connected in the ac side and L_2 in the rectified tension side (Fig. 1).

The circuit is made ideal in so far as it is assumed that the active resistance of inductors and the forward resistance of rectifiers is zero and the reverse resistance of rectifiers is infinite.

In [3] it was shown that the transient process (Fig. 2) consists of alternating excitation intervals, during which the rectified current rises, and switching intervals, during which the alternating tension pulse is spent in changing the magnitude and direction of the alternating current, but does not affect the magnitude of the rectified current, which remains unchanged in that interval. The duration of the intervals is changed in the course of the process. The part of the pulse spent in the ac circuit, without reaching the rectified current circuit, was called, in [3], the tension pulse switching loss.

This loss in the above circuit for (n + 1) half-cycles can be determined from the formula

$$p_k(n_t) = 2L_1 i(n_t), \tag{1}$$

where $i(n_t)$ is the instantaneous value of the rectified current at the beginning of the (n + 1)-th switching (all switchings begin at the moment the tension passes through zero).

In the above case Equation (1) represents the basic law of the transient process.

The current at the beginning of the (n + 1)-th switching is equal to the current rectified during the switching $i(n_t) = i_{II} = i_{II'} + i_{II''}$, where $i_{II'}$ and $i_{II''}$ are the currents in circuits I and II (Figs. 1 and 2).

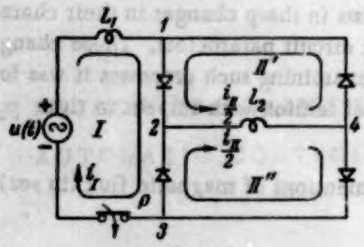

Fig. 1

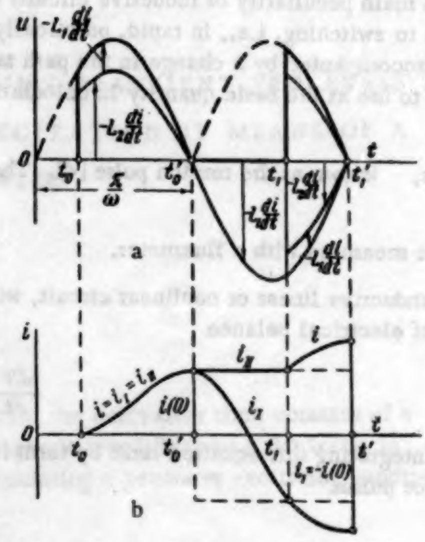

Fig. 2

During the (n + 1)-th half-cycle for a periodic function, or during the (n + 1)-th interval between two adjacent tension zeros in a general case, inductor L_2 accumulates an interlinkage increment $\Delta\Psi_2(n_t)$, whose value can be determined from expression

$$\Delta\Psi_2(n_t) = \frac{L_2}{L_2 + L_1}\,[p_u - 2L_1 i(n_t)]. \qquad (2)$$

Expression (2) can be easily interpreted. The bracket includes the difference between the full tension pulse p_u for the (n + 1)-th half-cycle and the switching loss of the pulse during the same period. The factor in front of the bracket determines the fraction of that difference which reaches the rectified current circuit.

The discrete time argument of difference $\Delta\Psi_2$ has an ordinal number a unit lower than that of the (n + 1)-th half-cycle under consideration, since $\Delta\Psi_2$ is considered to be an increment of function Ψ_2 at the end of the preceding half-cycle.

The value of the current $i(n_t) = i_{II}$ does not change during switching.

By substituting in (2) $\frac{\Psi_2(n_t)}{L_2}$ for $i(n_t)$, we obtain

$$\Delta\Psi_2(n_t) = \frac{L_2}{L_2 + L_1}\left[p_u - 2\,\frac{L_1}{L_2}\,\Psi_2(n_t)\right]. \qquad (3)$$

Expression (3) represents a finite difference equation. By substituting the first difference by its definition $\Delta\Psi_2(n_t) = \Psi_2(n_t + 1) - \Psi_2(n_t)$, we obtain a normal form of a difference equation

$$\Psi_2(n_t + 1) - \frac{L_2 - L_1}{L_2 + L_1}\,\Psi_2(n_t) = \frac{L_2}{L_2 + L_1}\,p_u. \qquad (4)$$

Since the phase and the moment t_0 of connecting the alternating tension to the circuit under consideration (Figs. 1 and 2) is arbitrary, the value of the initial pulse p_{u0} is also arbitrary in the limits $0 \le p_{u0} \le p_u$. The initial pulse p_{u0} determines the initial condition in the difference equation (4).

$$\Psi_2(0) = \frac{L_2}{L_2 + L_1}\,p_{u0}.$$

The solution of Equation (4), as is known, is obtained similarly to that of a linear differential equation with constant coefficients and represents the sum of a general solution of a homogeneous equation

$$\Psi_2(n_t + 1) - \frac{L_2 - L_1}{L_2 + L_1}\,\Psi_2(n_t) = 0$$

and a particular solution of the heterogeneous equation (4).

The general solution of the homogeneous equation is obtained in the form

$$\Psi_{2f}(n_t) = c_1 \lambda^{n_t}.$$

Subscript "f" in Ψ_{2f} stands for "free" and "c" for constrained.

Quantity λ_1 represents the root of the characteristic equation.

$$\lambda - \frac{L_2 - L_1}{L_2 + L_1} = 0.$$

Thus,

$$\Psi_{2f}(n_t) = c_1 \left(\frac{L_2 - L_1}{L_2 + L_1}\right)^{n_t}.$$

The particular solution of the heterogeneous equation (4) is found in the form of a constant $\Psi_2 = A$. By inserting this constant in (4) we obtain

$$A = \Psi_{2C} = \frac{L_2}{2L_1} p_u.$$

The general solution of Equation (4) is

$$\Psi_2(n_t) = \frac{L_2}{2L_1} p_u + c_1 \left(\frac{L_2 - L_1}{L_2 + L_1}\right)^{n_t}.$$

After inserting the initial condition, we obtain

$$c_1 = -L_1 \left(\frac{p_u}{2L_1} - \frac{p_{u0}}{L_2 + L_1}\right).$$

The solution of Equation (4) has the following form:

$$\Psi_2(n_t) = \frac{L_2}{2L_1} p_u - L_2 \left(\frac{p_u}{2L_1} - \frac{p_{u0}}{L_2 + L_1}\right)\left(\frac{L_2 - L_1}{L_2 + L_1}\right)^{n_t}. \tag{5}$$

By dividing both sides of Equation (5) by L_2 we obtain an expression for the instantaneous value of the rectified current i_{II} at the instants corresponding to the beginning of switching (zeros of tension)

$$i_{II}(n_t) = \frac{p_u}{2L_1} - \left(\frac{p_u}{2L_1} - \frac{p_{u0}}{L_2 + L_1}\right)\left(\frac{L_2 - L_1}{L_2 + L_1}\right)^{n_t}. \tag{6}$$

Denoting

$$\frac{L_2 - L_1}{L_2 + L_1} = e^{-\Delta\tau / T_{\text{equiv}}} \tag{7}$$

where $\Delta\tau = \frac{\pi}{\omega}$ and T_{equiv} is a constant, and $\omega = 2\pi f$ is the angular repetition frequency of function $u(t)$ zero values.

Let us call quantity T_{equiv} the equivalent time constant of an ideal bridge-type rectifying circuit with two inductors. By inserting (7) into (5), we obtain

$$\Psi_2(n_t) = \frac{L_2}{2L_1} p_u - L_2 \left(\frac{p_u}{2L_1} - \frac{p_{u0}}{L_2 + L_1}\right) e^{\frac{-n_t \Delta\tau}{T_{\text{equiv}}}} \tag{8}$$

The continuous exponential function

$$f_2(t) = \frac{L_2}{2L_1} p_u - L_2 \left(\frac{p_u}{2L_1} - \frac{p_{u0}}{L_2 + L_1}\right) e^{\frac{-\left(t - \frac{\pi}{\omega}\right)}{T_{\text{equiv}}}} \tag{9}$$

with the values of its argument $t = \frac{\pi}{\omega}, 2\frac{\pi}{\omega}, 3\frac{\pi}{\omega}, \ldots, (k+1)\frac{\pi}{\omega}$ has the same values as function $\Psi_2(n_l)$ with its argument $n_l = 0, 1, 2, \ldots, k$. Hence, the continuous function of time $f_2(t)$, which has continuous derivatives, can be used as an interpolating function for $\Psi_2(n_l)$ in the intervals between the points common to both functions.

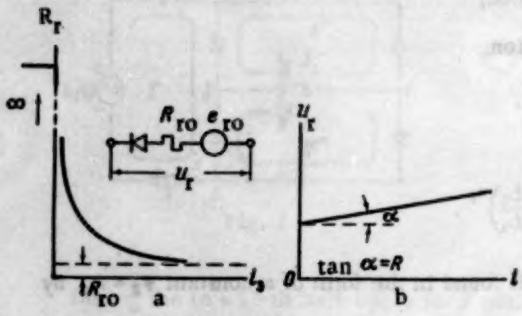

Fig. 3

For dealing with actual circuits an equivalent circuit is proposed in [3], in which the single-phase bridge-type circuit with semiconductor rectifiers is replaced by a representation of a real rectifier in the form of three components – a constant resistance, a source of constant back-emf and an ideal rectifier (Fig. 3)—and conditions of switching are examined. On the basis of analyzing switching conditions and observing by means of an oscillograph, it was shown in practice that the instant at which switching intervals begin in transient and steady-state processes coincides exactly with that at which the tension passes through zero, if in front of the rectifier (on the ac side) there is a choke coil with a saturating core, and there is a considerable linear inductance on the dc side. In this case the saturating choke simulates the magnetic amplifier and the considerable linear inductance the excitation winding of a machine.

2. Combined Operation of a Magnetic Amplifier and Rectifier in an Electrical Machine Excitation Circuit

The circuit under consideration consists of a self-magnetizing magnetic amplifier (MA) with an internal feedback, a single-phase bridge-type circuit with semiconductor rectifiers, an inductive load and a source of direct emf on the rectified current side. An equivalent circuit is shown in Fig. 4. The load parameters simulate the generator excitation winding and the resistance of the rectifiers and the operating winding MA, whereas the direct emf on the rectified current side represents a design value, determined by the value required to flatten out the excitation and rectifier characteristics obtained from curves u_{eo} in Fig. 5 and e_{ro} in Fig. 3 [3].

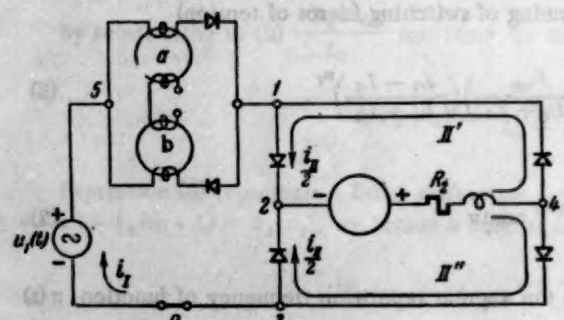

Fig. 4

In determining the equivalent active resistance R in the rectifying circuit it is necessary to keep in mind two essentially different cases.

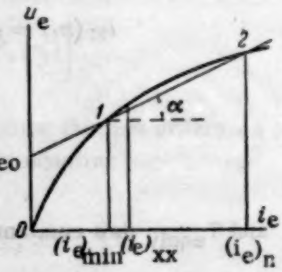

Fig. 5

Case 1. Resistance R_1 of the magnetic amplifier ac winding is small compared with resistance R of the excitation winding. This is the normal case.

Moreover, for low-voltage excitation the number of rectifier elements per mesh is small (one or two) and their resistance $R_{ro} = \tan\alpha$ (Fig. 3) is approximately equal to 0.5 ohm, which is small compared both with the active resistance R of the load and with the reactance of MA. In this case it becomes possible to neglect the active resistance in the ac circuit and take the equivalent active resistance R_2 as the sum of the excitation winding resistance R and the dc bridge resistance nR_{ro}. $R_2 = R + nR_{ro}$.

The equivalent emf E_2 is determined in this case by the expression $E_2 = E - 2ne_{ro}$, where E is the actual direct load emf or an equivalent quantity, measured by a length of curve u_{eo} in Fig. 5, n is the number of series elements per mesh, e_{ro} is the equivalent back-emf of a rectifier element (Fig. 3).

Case 2. The active resistance of the MA ac windings and the rectifiers in the ac circuit is comparable in magnitude with the active resistance of the rectified current circuit.

In this case in order to find an equivalent circuit the following simplifications can be made:

a) the resistance of the MA winding and the rectifiers can be transferred from the ac circuit to the bridge diagonal 2-4 (Fig. 4);

b) the rectifier design back-emf can be transferred to the bridge diagonal.

In this case the equivalent resistance is determined by the expression $R_2 = R_1 + (2n + n_{oc})R_{ro} + R_r$, where n_{oc} is the number of rectifying feedback elements in meshes 5-1 and 1-5 (Fig. 4), and the equivalent emf is determined by the expression $E_2 = E - (2n + n_{oc})\, e_{ro}$.

When an equivalent circuit was compiled in [3], use was also made of the operating peculiarities of rectifiers and magnetic amplifiers in the generator exciter excitation circuit, peculiarities conditioned by the existence of a considerable nonoperative exciter excitation current required for the generator nominal tension on open circuit. These peculiarities include: 1) a large switching interval; 2) a practically complete saturation of one of the magnetic amplifier cores during the excitation interval.

The latter peculiarity permits one to consider the magnetic amplifier dynamic inductance to be constant in the excitation interval, i.e., to consider the MA magnetizing characteristic to be linear above the saturation bend (Fig. 6).

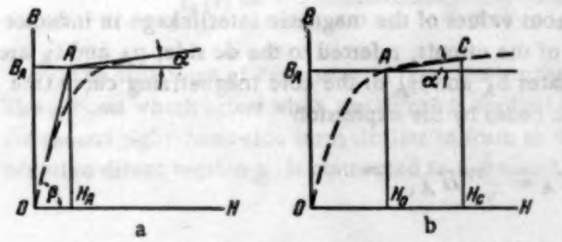

Fig. 6

Under conditions determined by these peculiarities, the magnetizing effect of the MA control circuit, as can be seen from the experimental curves, is practically independent of conditions in the MA operating winding or the inductive load which is connected in the dc side and simulates the exciter excitation winding. It is, therefore, possible to represent the MA control circuit as a linear directional first-order section (aperiodic section). The linearity control coefficients can be obtained experimentally as in [5] by means of investigating a model MA of the type required, of an arbitrary power and size, but working under actual load conditions. For magnetic amplifiers of the same type but a different power and size, paper [3] provides design formulas similar to those obtaining in the technical literature.

The basic aim of this paper is to investigate the transient and steady-state processes in the self-magnetizing MA inductive load, which is connected in the dc side of a single-phase bridge-type rectifier and which simulates the exciter excitation winding (Fig. 4).

Transients in the MA are caused by changes in the control current with a constant supply tension. This condition is typical for dc generators with magnetically controlled power and tension, when the controlling device is supplied from an independent ac source. It is also characteristic for the normal operation of synchronous generator magnetic regulators if the degree of static changes is small, when in the course of regulation supply-tension variations can be neglected, being small in comparison with changes in control current which is supplied from a highly sensitive measuring element.

The MA control circuit current deviations from the original value, which corresponds to the initial steady state, play in the above excitation circuit the role of a disturbing effect. In automatic regulation the excitation circuit is a sluggish link connected to the MA control circuit. In investigating the possibility of making the excitation circuit linear, both in the case of its analysis and for experimental work, the transient caused by a sudden change in the control current was taken as the basic form under consideration. When the circuit under investigation is fed by an alternating tension, such a transient process is superimposed on the preceding stable-state condition of the system (curve i_2 in Fig. 7).

The quantitative representation of transient and stable-state processes in the circuit under investigation (Fig. 4) was obtained by solving a linear difference equation with constant coefficients. The method of compiling a difference equation was similar to the one used for transients inthe ideal circuit case.

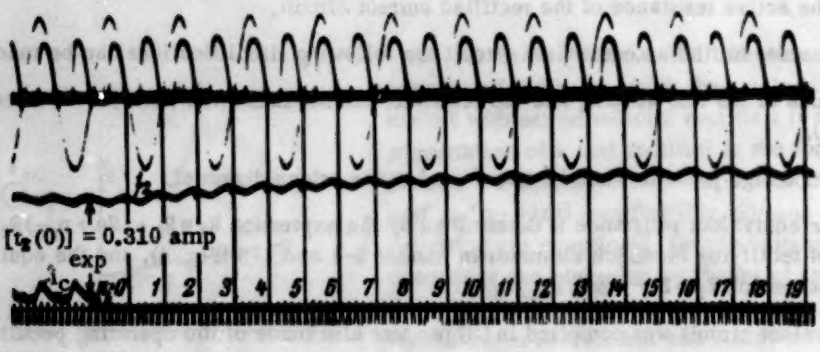

Fig. 7

We obtain the following difference heterogeneous linear equation with constant coefficients [3]:

$$\Psi_2(n_t+1) - \frac{\omega L_2 - \pi R_2}{\omega (L_1 + L_2)} \Psi_2(n_t) =$$
$$= \frac{L_2}{L_1 + L_2}\left(p_u + \frac{\pi}{\omega} E_2 - p_A - L_1 i_A + p_{ec}[n_t]\right). \tag{10}$$

In Equation (10) $\Psi_2(n_t + 1)$ and $\Psi_2(n_t)$ are instantaneous values of the magnetic interlinkage in inductor L_2 on the rectified current side, R_2 is the active resistance of the circuit, referred to the dc side, p_A and i_A are parameters of the MA magnetic system, related to coordinates B_A and H_A of the core magnetizing curve (the beginning of the linear characteristic beyond the saturation bend) by the expression

$$p_A = w_1 S_{st} B_A, \quad i_A = \frac{l_{st}}{w_1} H_A,$$

where w_1 is the number of turns in the MA operating winding, S_{st} is the cross section of the steel core, l_{st} is the length of the core magnetic path, L_1 is the constant inductance of the MA operating winding with a saturated core, $p_{ec}[n_t]$ is the equivalent control pulse for the n-th half-cycle, determined by the expression [3]

$$p_{ec}[n_t] = \frac{0.2\,\pi\mu_c S_{st} w_1 w_c}{l_{st}} i_c[n_t]. \tag{11}$$

In Expression (11) w_c is the number of turns in the MA control winding, $i_c[n_t]$ is the control current at the beginning of the n-th half-cycle, $\mu_c = \frac{l_{st}}{0.4\,\pi S_{st}} \gamma_1$ is the permeability for the constant component of the flux, γ_1 is the coefficient determined from the statistical characteristics $\Sigma\Delta\Phi_0 = f(i_c)$ which were obtained by means of a ballistic galvanometer or a fluxmeter for a given type magnetic amplifier with a magnetic circuit of length l_{st} and cross section S_{st} [5].

Current i_c in the control circuit is determined by the linear differential equation with constant coefficients [5]

$$E_c = R_c i_c + \gamma_1 w_c^2 \frac{di_c}{dt} 10^{-8}. \tag{12}$$

When an initial rectified current is followed by a sudden reduction in the control current from a certain value to zero, the solution of the difference equation (10) assumes the following form:

$$\Psi_2(n_t) = L_2 \frac{p_u + \frac{\pi}{\omega}E_2 - p_A + L_1 i_A}{\frac{\pi}{\omega}R_2 + L_1} + L_2 \frac{p_{ec}[0]}{\frac{\pi}{\omega}R_2 + L_1}\left(\frac{\omega L_2 - \pi R_2}{\omega L_1 + \omega L_2}\right)^{n_t}, \tag{13}$$

where E_2 is the equivalent emf on the rectified current side and is determined when the characteristic of the exciter and rectifier is made linear; it is included in the initial conditions of the process.

When the supply tension is sinusoidal the expression for instantaneous values of the rectified current at the instants the tension passes through zero takes the following form:

$$i_2(n_t) = \frac{0{,}9\,u_{\text{eff}} + E_2 - 2f(p_A - L_1 i_A)}{R_2 + 2fL_1} + \frac{2fp_{\text{ec}}[0]}{R_1 + 2fL_1}\left(\frac{L_2 - \frac{1}{2f}R_2}{L_2 + L_1}\right)^{n_t}. \tag{14}$$

The continuous exponential function

$$i_2(t) = \frac{0{,}9\,u_{\text{eff}} + E_2 - 2f(p_A - L_1 i_A)}{R_2 + 2fL_1} + \frac{2fp_{\text{ec}}[0]}{R_2 + 2fL_1} e^{-\frac{t}{T_{\text{equiv}}}}, \tag{15}$$

where

$$T_{\text{equiv}} = \frac{1}{2f \ln \dfrac{L_2 + L_1}{L_2 - \frac{1}{2f} R_2}}, \tag{16}$$

coincides with function $i_2(n_t)$ at points $t = 0, \frac{\pi}{\omega}, 2\frac{\pi}{\omega}, \ldots, n\frac{\pi}{\omega}$ and can serve as an interpolating function in the intervals between these points. Expression (15) can be represented more conveniently in the form

$$i_2(t) = \frac{0{,}9\,u_{\text{eff}} + E_2 - 2f(p_A - L_1 i_A) + 2fp_{\text{ec}}[0]}{R_2 + 2fL_1} - \frac{2fp_{\text{ec}}[0]}{R_2 + 2fL_1}\left(1 - e^{-\frac{t}{T_{\text{equiv}}}}\right). \tag{17}$$

The first term of Expression (17) represents the stable-state current before the application of the signal. The process which arises when the signal is applied (when the MA control circuit is opened) is represented by the second right-hand-side term, similar in form to the function which represents the transient process when a negative direct tension u is connected to a linear L, R circuit:

$$i = -\frac{u}{R}\left(1 - e^{-\frac{t}{T}}\right).$$

Let us examine more closely the second term of Expression (17).

When an analysis is made of the process due to breaking the control circuit in which a demagnetizing current is flowing, it is necessary to take into account the negative sign of the "signal" $p_{ec}(0)$, i.e., examine a positive function

$$i = \frac{2fp_{\text{ec}}[0]}{R_2 + 2fL_1}\left(1 - e^{-\frac{t}{T_{\text{equiv}}}}\right). \tag{18}$$

Switching on a positive constant magnetization is equivalent to switching off a negative magnetization. Hence, Expression (18) also represents the process of applying a constant positive signal $2fp_{ec}(0) = \text{const.}$

Let us find a differential equation whose solution is function (18). Let us find it in the form

$$A\frac{di}{dt} + Bi = \frac{2fp_{\text{ec}}[0]}{R_2 + 2fL_1}. \tag{19}$$

Substituting current i, determined by Expression (18), and its derivative $\frac{di}{dt}$ in Equation (19) and equating the coefficients of the right and left sides of the equation we shall obtain $B = 1$, $A = BT_{\text{equiv}} = T_{\text{equiv}}$.

Equation (19) will then become

$$T_{\text{equiv}}\frac{di}{dt} + i = \frac{2fp_{\text{ec}}[0]}{R_2 + 2fL_1}. \tag{20}$$

Taking into consideration (11) we can write

$$T_{\text{equiv}}\frac{di}{dt} + i = k_1 i_c, \qquad (21)$$

where

$$k_1 = \frac{0{,}4\pi\mu_c S_{st} w_1 w_c f}{l_{st}(R_2 + 2fL_1)}.$$

The magnitude of current i_c, as it has already been pointed out, is determined by Equation (12).

Expressions (19) and (21) are equations of first-order sections (aperiodic).

Thus, the circuit under investigation (Fig. 4) can be represented, when used as a part of an automatic control circuit, by a combination of two directional aperiodic sections.

In the case when the automatic magnetic excitation regulator has no sluggish measuring element,* the regulator consists of the two above-mentioned inert sections. In automatic control the exciter is always connected to the very sluggish circuit of the generator excitation winding, which almost completely smoothes out the exciter tension pulsations, so that any further consideration of closed automatic excitation control circuits which include a magnetic regulator can be carried out on the basis of known methods of the automatic control theory.

Above analytical deductions, in particular Expression (15), were verified by a large number of experiments. The results of one of them are given on the oscillogram in Fig. 7, obtained with $T_{\text{equiv}} = 0{,}0992$ sec. Circles denote the calculated points of the theoretical curve, which in this case coincides with the experimental one.

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* Such regulators include, for instance, automatic magnetic locomotive power regulators, which have a tachometer generator as a measuring element.

SYNCHRONOUS REACTIVE MOTOR SPEED REGULATION IN SYSTEMS OF PRECISE MAGNETIC RECORDING

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(Moscow)

The author analyzes the stability of speed control by a synchronous reactive motor in a system of accurate magnetic recording. The stability conditions are obtained for the case when an electronic or electromechanical phase discriminator is used as the sensing element.

The use of a nonperforated magnetic tape as a magnetic memory storage unit is complicated, in many cases, by the fact that the tape is not a perfectly elastic body and suffers changes in dimensions with the passage of time. The variable tension to which the tape is subjected during recording and subsequent playback produces permanent deformations varying in magnitude along the length of the tape. This results in the recording frequency coinciding only occasionally with the frequency of the reproducible signal, and even then only for a very short length of time. Figure 1 shows an oscillogram of the beat note between a highly stable frequency, recorded on tape, and the same signal during its playback from the tape.

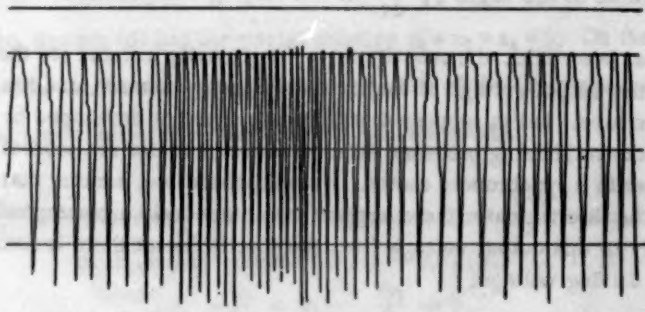

Fig. 1.

It follows from Fig. 1 that the time scale experiences appreciable distortion during the reproduction of some information recorded on tape. This distortion in the time scale can be tolerated in sound reproduction, but for many forms of recording (phototelegraph and video signals, telemetered data, by means of frequency modulation) they are absolutely inadmissible. Consequently, in all known arrangements for accurate magnetic recording on a nonperforated tape, automatic control is applied to the mean tape speed, thereby achieving an almost complete compensation for all changes in the time scale which occur on the tape [1-5].

The controlled element in most systems of accurate magnetic recording is a synchronous motor which keeps the tape in continuous motion through the tape winder [1-3, 5]. Figure 2 shows the block schematic of the automatic speed control system for the tape drive synchronous motor. A control signal, in the form of a constant-frequency sinusoidal voltage is recorded, simultaneously with the information signal, on the tape.

During reproduction the phase of the reproducible signal is compared with the phase of a signal, of the same frequency, from the standard generator a. The output of the phase detector b is applied to a variable-frequency signal generator c, thereby changing its output frequency. The output of the signal generator is amplified and applied to the synchronous motor which drives the tape in the reproducer. Information given in foreign literature [1, 3, 5] indicates great accuracy in systems similar to the one described above. Thus, in the magnetic delay system installed in Singapore, during the XVI Olympic Games in Melbourne, for the reception of phototelegraph signals, the error in maintaining the mean speed, during the time of transmission of one blank of phototelegraph information, did not exceed 10^{-5}.

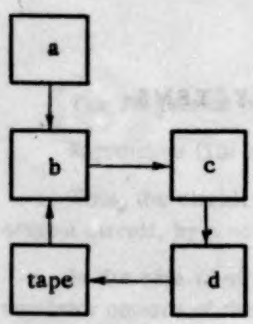

Fig. 2

In the present work, the stability of automatic speed control of a synchronous motor is analyzed and recommendations are made for the choice of the parameters of this system. Since magnetic recording systems incorporate almost exclusively synchronous reactive motors, all further discussions refer to such a type of motor.

Stability Conditions for the Control Process

The speed control of a synchronous motor must take place in such a way that the rotor is at no time out of synchronism, i.e., that the angle φ, equal to twice the angle between the axes of the stator and rotor poles, never exceeds $\varphi_{max} \approx \pi/2$ [6]. It follows from this that the rate of change of the frequency of the driving voltage can never exceed a known limit, determined by the rotor's mechanical time constant. Leaving the question of the value of this limit open, let us suppose that changes in the time scale resulting from a nonuniform deformation of the tape take place so slowly that the motor does not go out of synchronism during the whole control process.

Then, after making all simplifying assumptions, which are usually made during the analysis of the transient processes in a synchronous reactive motor [6], the rotor's equation of motion can be written in the form

$$\frac{d^2\varphi}{dt^2} + r\frac{d\varphi}{dt} + \omega_0^2 \sin\varphi = \omega_0^2 \sin(\varphi_0 + \psi). \tag{1}$$

Here φ_0 is the stable value of the angle φ, $\frac{d\psi}{dt}$ is the varying component of the drive frequency.

On the basis of what has been said, the quantity ψ is functionally related to the phase difference between two signals, one of which is the output voltage of the standard signal generator, and the other is taken from the control track of the magnetic tape. Let us suppose that the error signal is developed by a conventional phase discriminator. The rates of the stabilizing processes in such a discriminator are several orders higher than the rates of the transient processes in a synchronous motor. We will, therefore, assume that for every phase difference that appears at the input to the discriminator there appears instantaneously a corresponding voltage at its output. Corresponding instantaneously to this output voltage from the discriminator, there is some value of the varying-frequency component of the driving voltage.

Assuming that the control system is linear and that the time scale variations occurring along the tape are sinusoidal, we can write

$$\frac{d\psi}{dt} = a\cos\Omega t + q(\varphi - \varphi_0). \tag{2}$$

Here Ω is the angular frequency of the sinusoidal variations in the time scale, $\varphi - \varphi_0$ is the phase difference determined by the presence of transients in the motor.

From (1) and (2), we obtain

$$\frac{1}{q}\frac{d^3\psi}{dt^3} + \frac{a}{q}\Omega^2\cos\Omega t + \frac{r}{q}\frac{d^2\psi}{dt^2} + \frac{a}{q}r\Omega\sin\Omega t + \omega_0^2\sin\left[\frac{1}{q}\frac{d\psi}{dt} + \varphi_0 - \frac{a}{q}\cos\Omega t\right] = \omega_0^2\sin[\varphi_0 + \psi]. \tag{3}$$

Setting $\frac{a}{q} = b\mu$, we rewrite (3) in the form

$$\frac{d^3\psi}{dt^3} + r\frac{d^2\psi}{dt^2} + \omega_0^2 q \sin\left[\frac{1}{q}\frac{d\psi}{dt} + \varphi_0 - b\mu\cos\Omega t\right] = \tag{4}$$

$$= \omega_0^2 q \sin(\varphi_0 + \psi) - qb\mu\Omega^2\cos\Omega t - b\mu qr\Omega\sin\Omega t$$

or

$$\frac{d^3\psi}{dt^3} + r\frac{d^2\psi}{dt^2} - \omega_0^2 q \sin(\varphi_0 + \psi) + \omega_0^2 q \sin\left[\frac{1}{q}\frac{d\psi}{dt} + \varphi_0\right] = \tag{5}$$

$$= \frac{\mu bq\omega_0^2}{2}\cos\left[\frac{1}{q}\frac{d\psi}{dt} + \varphi_0\right]\cos\Omega t - \mu qb\Omega^2\cos\Omega t -$$

$$- \mu bqr\Omega\sin\Omega t + \mu^2 F\left(\frac{d\psi}{dt}, t, \mu\right).$$

Here $F\left(\frac{d\psi}{dt}, t, \mu\right)$ is a periodic function of time t with period $\frac{2\pi}{\Omega}$.

Taking μ to be a small parameter, we replace Equation (5) by a series of first-order equations:

$$\frac{dx_1}{dt} = -rx_1 - \omega_0^2 q \sin\left[\frac{1}{q}x_2 + \varphi_0\right] + \omega_0^2 q \sin[x_3 + \varphi_0] +$$

$$+ \frac{\mu bq\omega_0^2}{2}\cos\left[\frac{1}{q}x_2 + \varphi_0\right]\cos\Omega t - \mu qb\Omega^2\cos\Omega t -$$

$$- \mu bqr\Omega\sin\Omega t + \mu^2 F(x_2, t, \mu), \quad \frac{dx_2}{dt} = x_1, \tag{6}$$

$$\frac{dx_3}{dt} = x_2,$$

where $x_3 = \psi$.

For μ equal to zero, System (6) has the trivial solution $x_1 = x_2 = x_3 = 0$. On the basis of this we shall seek a solution of Equations (6) in the form $x_i = \mu y_i$. Expanding each of the functions $\sin\left[\frac{1}{q}x_2 + \varphi_0\right]$, $\cos\left[\frac{1}{q}x_2 + \varphi_0\right]$ and $\sin[x_3 + \varphi_0]$ into Maclaurin series about the point $x_i = 0$, we obtain

$$\frac{dy_1}{dt} = -ry_1 - \omega_0^2\cos\varphi_0 y_2 + \omega_0^2 q\cos\varphi_0 y_3 + \frac{bq\omega_0^2}{2}\cos\varphi_0\cos\Omega t -$$

$$- qb\Omega^2\cos\Omega t - bqr\Omega\sin\Omega t + \mu F_1(y_2, y_3, t, \mu), \tag{7}$$

$$\frac{dy_2}{dt} = y_1, \quad \frac{dy_3}{dt} = y_2.$$

The following assertion is associated with systems of equations of the form of (7). If the equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{nn} - \lambda \end{vmatrix} = 0, \tag{8}$$

does not have any roots with their real parts equal to zero, then the system of the form (7) permits, for sufficiently small μ, one and only one almost periodic solution which reduces, for $\mu = 0$, to a generating one (i.e., to a solution of the system in (7) with $\mu = 0$) [7]. In Equation (8) a_{ik} are the constant coefficients associated with the first-degree unknowns y_i in the right side of equations of the form of (7). In our case, the time-dependent right sides of Equations (7) appear as periodic functions of t with period $2\pi/\Omega$. Thus, on the basis of the assertion presented above, for the case of the absence in Equation (8) of roots with zero real parts, the system of equations

in (7) has only one periodic solution with period $2\pi/\Omega$, which changes, for $\mu = 0$, to a generating one. This is equivalent to the assertion regarding the existence of a periodic solution for System (6), with the solution becoming identically zero for $\mu = 0$. Let us find the stability conditions for this solution. First of all, let us note that the right sides as well as the periodic solution of System (7) are analytic functions of μ.

Let us form the following equations for System (7):

$$\frac{d\varepsilon_i}{dt} = \sum p_{ik} z_k \, (i, k = 1, 2, 3). \tag{9}$$

It is not difficult to see that p_{ik} are periodic functions having period $2\pi/\Omega$ and are analytic with respect to μ, and that, for $\mu \to 0$ any of the p_{ik} tends to the corresponding coefficient of the first-degree unknown in the equations of System (7).

Liapunov's Theorem states that if all the characteristic exponents of unperturbed periodic motion have negative real parts, then this motion is asymptotically stable [7]. On the other hand, the coefficients of the characteristic equation of a system such as (9) will be analytic functions of μ and will become, for $\mu = 0$, the coefficients corresponding to the characteristic equation of a system such as (9), in which the p_{ik}'s are replaced by their limiting values for $\mu \to 0$, i.e., a_{ik}. In this way, the condition for the existence and for the asymptotic stability of the periodic solution of System (7), for sufficiently small μ, is the existence of negative real parts in all the roots of the fundamental equation (8).

In our case, $a_{11} = -r$, $a_{12} = -\omega_0^2 \cos\varphi_0$, $a_{13} = \omega_0^2 q \cos\varphi_0$, $a_{21} = 1$, $a_{22} = 0$, $a_{23} = 0$, $a_{31} = 0$, $a_{32} = 1$, and $a_{33} = 0$.

The corresponding Hurwitz determinants are equal to:

$$\Delta_1 = r, \ \Delta_2 = r\omega_0^2 \cos\varphi_0 + \omega_0^2 q \cos\varphi_0, \quad \Delta_3 = -\omega_0^2 q \cos\varphi_0 \Delta_2.$$

It follows from here that the stability condition is $\omega_0 q \cos\varphi_0 < 0$, and since $\varphi < \frac{\pi}{2}$ then $q < 0$. Denoting $|q| = p$, we have for the second condition the inequality $r > p$.

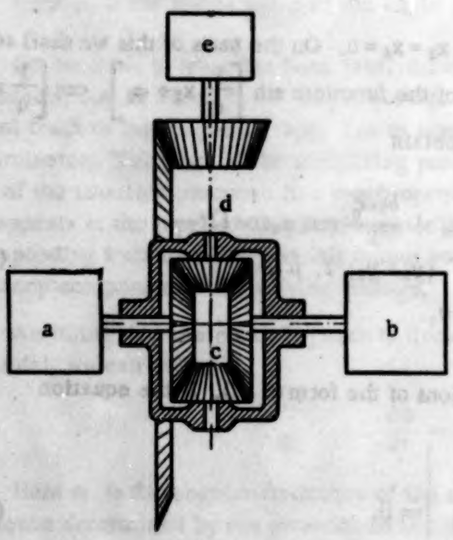

Fig. 3

In most foreign equipment, designed for accurate magnetic recording on nonperforated tape, the phase detector is usually a system such as the one shown schematically in Fig. 3 [1 - 3]. Here a and b are two synchronous motors with their rotors rigidly coupled to two shafts of the differential c. The differential's third shaft d, the angle of rotation of which is equal to the difference between the angles of rotation of the first two shafts, is rigidly coupled to the potentiometer e or to a system of selsyns which operate as transformers.

The stator winding of one motor is fed by a voltage taken from the output terminals of the standard oscillator, while that of the second motor is fed by the voltage from the control track of the magnetic tape. In the presence of a phase difference between these two voltages, the angles of rotation of the rotors are unequal and hence the shaft d turns through a certain angle and thereby changes the output voltage from the potentiometer e. This leads to a change in the driving voltage's frequency and the speed of rotation of the motor driving the tape. The equation of motion of this motor's rotor can, as before, be written in the form

$$\frac{d^2\varphi}{dt^2} + r\frac{d\varphi}{dt} + \omega_0^2 \sin\varphi = \omega_0^2 \sin(\varphi_0 + \psi). \tag{10}$$

The equation of motion for the rotor of the motor which is fed by the voltage derived from the control track will have the form

$$\frac{d^2\chi}{dt^2} + r'\frac{d\chi}{dt} + \omega_0'^2 \sin\chi = \omega_0'^2 \sin(\chi_0 + \beta), \tag{11}$$

where χ and β are quantities of the same type as φ and ψ.

Since the motor driven by the standard voltage is not included in the control feedback loop then, assuming that its rotor is free from the influence of any disturbances, we can assume the rotor's rotation to be strictly uniform. In other words, the rotating rotor of this motor appears as an original metering system, with respect to which the rotor connected to the other shaft of the differential rotates. The constant component of the relative angle of rotation of both rotors corresponds to some nominal value of the frequency from the driving signal generator which, in turn, is equal to the nominal frequency of the reproducible signal. Each change in the relative angle of rotation occurs with a corresponding additional turning of the rotor of the motor, the driving voltage for which is taken from the control track. Hence, we can write $\frac{d\psi}{dt} = n(\chi - \chi_0)$, where $\frac{d\psi}{dt}$ is the varying-frequency component of the voltage driving the tape-transport motor.

On the other hand, $\beta = a\cos\Omega t + b(\varphi - \varphi_0)$. Let us set $\int(\chi - \chi_0)\,dt = \epsilon$. Equations (10) and (11) can then be rewritten in the form

$$\frac{d^2\beta}{dt^2} + r\frac{d\beta}{dt} + b\omega_0^2 \sin\left[\frac{\beta}{b} + \varphi_0 - \frac{a}{b}\cos\Omega t\right] - b\omega_0^2 \sin[\varphi_0 + n\varepsilon] = \tag{12}$$
$$= -a\,\Omega^2 \cos\Omega t - a\,\Omega \sin\Omega t,$$
$$\frac{d^3\varepsilon}{dt^3} + r'\frac{d^2\varepsilon}{dt^2} + \omega_0'^2 \sin\left[\frac{d\varepsilon}{dt} + \chi_0\right] = \omega_0'^2 \sin[\chi_0 + \beta].$$

Let us set $a = \mu c$ and assume μ to be a small parameter. Equation (12) can then be replaced by the system equations

$$\frac{dx_5}{dt} = -r'x_5 - \omega_0'^2 \sin[x_4 + \chi_0] + \omega_0'^2 \sin[\chi_0 + x_1],$$
$$\frac{dx_4}{dt} = x_5, \qquad \frac{dx_3}{dt} = x_4,$$
$$\frac{dx_2}{dt} = -rx_2 - b\omega_0^2 \sin\left[\frac{x_1}{b} + \varphi_0 - \frac{\mu c}{b}\cos\Omega t\right] + b\omega_0^2 \sin[\varphi_0 + nx_3] - \tag{13}$$
$$-\mu c\Omega^2 \cos\Omega t - \mu c\Omega \sin\Omega t,$$
$$\frac{dx_1}{dt} = x_2, \quad x_1 = \beta, \quad x_3 = \varepsilon.$$

Let us rewrite Equation (13) in the form

$$\frac{dx_5}{dt} = -r'x_5 - \omega_0'^2 \sin[x_4 + \chi_0] + \omega_0'^2 \sin[\chi_0 + x_1],$$
$$\frac{dx_4}{dt} = x_5, \quad \frac{dx_3}{dt} = x_4, \quad \frac{dx_2}{dt} = -rx_2 - b\omega_0^2 \sin\left[\frac{x_1}{b} + \varphi_0\right] +$$
$$+ \frac{\mu c\omega_0^2}{2}\cos\left[\frac{x_1}{b} + \varphi_0\right]\sin\Omega t + b\omega_0^2 \sin[\varphi_0 + nx_3] - \tag{14}$$
$$-\mu c\Omega^2 \cos\Omega t - \mu c\Omega \sin\Omega t + \mu^2 F(x_1, t, \mu),$$
$$\frac{dx_1}{dt} = x_2,$$

where $F(x_1, t, \mu)$ is a periodic function of t, with period $\frac{2\pi}{\Omega}$, and is analytic with respect to the parameter μ.

For $\mu = 0$, System (14) has the trivial solution $x_i = 0$. All the above discussions, concerning System (6) are also applicable to System (14). The condition for the existence and for the asymptotic stability of the periodic solution for System (14) will be the absence of roots, with positive or zero real parts, in the fundamental equation

$$
\begin{vmatrix}
a_{11}-\lambda & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22}-\lambda & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33}-\lambda & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{44}-\lambda & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}-\lambda
\end{vmatrix} = 0.
$$

Here $a_{ik} = \left(\frac{\partial X_i}{\partial x^k}\right)_{[x_1=x_2=x_3=x_4=x_5=0,\ \mu=0]}$, and X_i are the right halves of the system of equations (14).

For the case under consideration: $a_{11} = a_{13} = a_{14} = a_{15} = 0,\ a_{12} = 1,\ a_{21} = -\omega_0^2 \cos\varphi_0,\ a_{22} = -r,\ a_{23} = bn\omega_0^2\cos\varphi_0,\ a_{24} = 0,\ a_{25} = 0,\ a_{31} = a_{32} = a_{33} = a_{35} = 0,\ a_{34} = 1,\ a_{41} = a_{42} = a_{43} = a_{44} = 0,\ a_{45} = 1,\ a_{51} = \omega_0'^2\cos\chi_0,\ a_{52} = 0,$
$$a_{53} = 0,\ a_{54} = -\omega_0'^2 \cos\chi_0,\ a_{55} = -r'.$$

The fundamental equation will be written in the form

$$
\begin{aligned}
&\lambda^5 + \lambda^4 [r + r'] + \lambda^3 [rr' + \omega_0'^2 \cos\chi_0 + \omega_0^2 \cos\varphi_0] + \\
&+ \lambda^2 [r\omega_0'^2 \cos\chi_0 + r'\omega_0^2 \cos\varphi_0] + \lambda\omega_0^2\omega_0'^2 \cos\varphi_0 \cos\chi_0 - \\
&- \omega_0^2\omega_1'^2 bn \cos\varphi_0 \cos\chi_0 = 0.
\end{aligned} \tag{15}
$$

From Hurwitz's conditions it follows, firstly, that $-\omega_0^2\omega_0'^2\, bn \cos\varphi_0 \cos\chi_0 > 0$ and, since $\varphi_0 < \frac{\pi}{2}$, $\chi_0 < \frac{\pi}{2}$, it must be that $bn < 0$.

The second Hurwitz condition $\Delta_2 > 0$ has the form

$$(r + r')[rr' + \omega_0'^2 \cos\chi_0 + \omega_0^2 \cos\varphi_0] > r\omega_0'^2 \cos\chi_0 + r'\omega_0^2 \cos\varphi_0. \tag{16}$$

This condition is satisfied identically for $r \neq 0,\ r' \neq 0$.

Hurwitz's third condition $\Delta_3 > 0$ leads to the inequality

$$
\begin{aligned}
&[r\omega_0'^2 \cos\chi_0 + r'\omega_0^2 \cos\varphi_0](r + r')[rr' + \omega_0'^2 \cos\chi_0 + \omega_0^2 \cos\varphi_0] + \\
&+ (r + r')\,\omega_0'^2\omega_0^2 \cos\varphi_0 \cos\chi_0 > [r\omega_0'^2 \cos\chi_0 + r'\omega_0^2 \cos\varphi_0]^2 + \\
&+ (r + r')\,\omega_0^2\omega_0'^2\, |bn| \cos\varphi_0 \cos\chi_0.
\end{aligned} \tag{17}
$$

Finally, Hurwitz's fourth condition is written in the form

$$
\begin{aligned}
&[(r + r')\,rr' + r\omega_0^2 \cos\varphi_0 + r'\omega_0'^2 \cos\chi_0]\,[r\omega_0'^2 \cos\chi_0 + r'\omega_0^2 \cos\varphi_0 + \\
&+ bn\,(rr' + \omega_0'^2 \cos\chi_0 + \omega_0^2 \cos\varphi_0)] > \\
&> \omega_0^2\omega_0'^2 \cos\varphi_0 \cos\chi_0^2\, [(r + r') + bn]^2.
\end{aligned} \tag{18}
$$

Inequalities (17) and (18) impose definite conditions on the values of the coefficients $\underline{b}$ and $\underline{n}$. To determine these conditions let us carry out a certain transformation of the inequalities (17) and (18). As a result of this transformation both inequalities, which must be satisfied by the absolute value of the product $|bn|$, reduce to the inequalities

$$
\begin{gathered}
|bn| < 1 + \frac{A}{BD}, \\
D\,|bn|^2 + (C - 2DB)\,|bn| + DB^2 - A < 0.
\end{gathered} \tag{19}
$$

Here

$$A = [(r + r')\,rr' + r\omega_0^2 \cos\varphi_0 + r'\omega_0'^2 \cos\chi_0][rr' + r'\omega_0'^2 \cos\chi_0 + \omega_0^2 \cos\varphi_0],$$

$$B = (r + r'),$$

$$C = [(r + r')\,rr' + r\omega_0^2 \cos\varphi_0 + r'\omega_1'^2 \cos\chi_0]\,[rr' + \omega_0'^2 \cos\chi_0 + \omega_0^2 \cos\varphi_0],$$

$$D = \omega_0^2\omega_0'^2 \cos\varphi_0 \cos\chi_0.$$

Since for $x \to -\infty$ and $x \to \infty$ the three-term equation $Dx^2 + (C - 2DB)\,x + (DB^2 - A) \to +\infty$, then for the second of the inequalities in (19) to be satisfied it is necessary that the following relation be satisfied:

$$x_1 < |bn| < x_2, \tag{20}$$

where x_1 and x_2 are the roots of the equation $Dx^2 + (C - 2\,DB)\,x + (DB^2 - A) = 0$.

The use of the electromechanical phase-sensitive device discussed, instead of a simple electrical phase-discriminator circuit, has the following advantages:

1) although the introduction of extra degrees of freedom complicates the control system, nevertheless, this makes it more flexible, i.e., it increases the freedom in the choice of values of the separate parameters for the provision of stability of the control process;

2) any of the purely electrical phase-discriminator circuits impose very high requirements on the stability of the output voltage, in the absence of an error, since a given relative change in the magnitude of this voltage will bring about a corresponding relative change in the output frequency of the signal generator.

In the case of the phase-sensitive electromechanical device, any change in the voltage driving each of the motors will bring about the appearance of a disturbing moment acting on the corresponding rotor. The hunting of this rotor will lead to a relative change in frequency produced by the signal generator. However, this change will be smaller relative to the voltage change.

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INPUT CIRCUITS OF CONTACT-MODULATED AMPLIFIERS

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The main requirements for the input circuits of electronic amplifiers in automatic potentiometers are set forth. A table of diagrams of the input circuits, containing contact modulators, is given, as well as the formulas for determining the parameters of these circuits. The features of each circuit are set forth and recommendations are made as to their application in amplifiers of automatic potentiometers.

Wide application has been found in automatic control equipment of amplifiers in which an incoming dc signal is converted to an alternating one at the input. The dc-to-ac conversion results in a considerable reduction of the zero-level drift, and thus, overcomes the basic shortcoming of dc amplifiers. Amplifiers with dc-to-ac conversion of the input signal find application in various systems of automatic control, tracking systems, automatic potentiometers, modulators, and other equipment where low-level dc voltages are required (of the order of a millivolt or less). In most cases the input circuits of such amplifiers contain contact modulators possessing the smallest drifts among those presently known [1-4].

Many circuits, each one unique in a different way, exist for connecting the contact modulator. The proper choice of the input circuit and its components is governed, to a great extent, by the amplifier's characteristics and the system as a whole.

In the literature of this country there have been no papers published on the comparison analysis of the indicated systems. In this article the results of experiments carried out on the more common input circuits, containing contact modulators, are given, and recommendations are made regarding their application in amplifiers of automatic potentiometers.

A series of special requirements is imposed on the input circuits of amplifiers used in automatic potentiometers. The more essential ones for the analysis and evaluation of the circuits being studied are listed below.

The input circuit must convert the incoming dc signal to an ac signal at the network frequency. At the same time it is expedient to have the largest possible conversion factor (ratio of the effective magnitude of the fundamental at the output to the direct signal at the input). Insofar as the input signal contains, in most cases, a significantly large ac component of the network frequency, it is desirable for the input circuit to discriminate against, or significantly attenuate, the indicated alternating component. In order to prevent overloading of the source and yet obtain maximum sensitivity, the input resistance of the circuit must be at least several times larger than the output impedance of the measuring circuit. It is imperative that the disturbances, introduced by the input circuit (among them the zero drift), possess minimum values and do not extend into the regions of the equipment's sensitivity. The input circuit must be as insensitive as possible to the action of external electric and magnetic fields. If the amplifier is designed for operation in fast-response systems, then the dynamic properties of the input circuit must be such that the period of the transient process* is not greater than 3-4 periods

* By period of the transient process is understood the time for the output to reach 98% of its steady-state value.

of the conversion frequency, i.e., 0.06-0.08 sec [1]. In those cases where, for some reason or other, the input to the amplifier cannot be grounded, the input unit must provide for the isolation of the input circuit.

There are many varieties of input circuits containing contact modulators. Table 1 lists the more common circuits and the corresponding equations for their basic parameters: the conversion factor K_p, input resistance R_{in}, mean input resistance $R_{in\,mean}$, and the selectivity factor α.

Equations for α are given for the case $R_i = 0$. In the table, φ is the phase angle of the input signal relative to the phase shift of the contact modulator, L_2 is the inductance of the transformer secondary winding, R is the total transformer loss resistance referred to the secondary winding, Q is the quality factor of the transformer, R_i is the internal resistance of the signal source.

The derivation of the equations for most of the parameters of the circuits in Table 1 is very lengthy (see [1]) and is not given here. Let us, therefore, limit ourselves to a few explanations.

The parameters of transformerless circuits, (1 to 8 in the table), were derived on the assumption that $2f_0C_cR_c \gg 1$ (f_0 is the modulation frequency) and the contact-travel time is zero. Moreover, the intercontact capacitances as well as the capacitances of the contacts to ground were ignored, this being permissible for input resistances up to 10 meg when $f_0 = 50$ cycles.

Under the above assumptions one can assume that the output signal consists of rectangular pulses. To determine K_p, the pulse amplitudes were obtained, whereupon the output signal was expressed in terms of a Fourier series, and the fundamental component, together with K_p, was determined.

R_{in} was derived on the basis of the condition that for $R_i = R_{in}$ the transfer coefficient K_p is reduced by a factor of two. Knowing R_{in} permits one to find K_p for a given source internal resistance R_i.

The mean value of the input resistance $R_{in\,mean}$ was determined from the relationship between the input voltage and the input current, averaged over one period. $R_{in\,mean}$ characterizes the loading of the source by the input circuit. The selectivity coefficient α was determined from the relationship between the conversion coefficients for dc signals (useful signal) and ac signals (conversion frequency).

It is apparent from the table that circuits 6 and 8 possess the largest K_p, R_{in}, and $R_{in\,mean}$ among the transformerless networks. Circuits 3, 4, and 7 possess slightly poorer parameters; however, their advantage rests in the fact that one modulator contact is free and can, in some cases, be utilized for other purposes, such as a demodulator. Circuits 1, 2, and 5, have one serious shortcoming: the absence of a decoupling capacitor results in the added dependence of the zero drift on the grid current of the input tube. This makes the application of these circuits in sensitive amplifiers highly undesirable. The transformerless circuits 1-6 possess one common disadvantage: their selectivity coefficient is very close to unity. As a result, considerable error will be introduced by the parasitic conversion-frequency input signal. If these circuits are utilized, the alternating component of the input signal must be suppressed to a level below the sensitivity threshold. The reduction of the alternating component necessitates the use of filters which have a very appreciable detrimental effect on the dynamic properties of the system. Therefore, in the presence of an appreciable alternating component in the input, it is expedient to use circuits 7 and 8 which are free of the above-mentioned shortcoming [6]. Because the parasitic alternating signal is delivered both to the grid and the cathode of the input tube, the fundamental component in the output is attenuated. If the circuit parameters are chosen so that the following conditions hold:

for circuit 7

$$\frac{1}{R_1} = \frac{\mu}{\mu+2}\left(\frac{1}{R_2} + \frac{1}{R_k}\right) \tag{1}$$

and for circuit 8

$$R_d = R_k\left(1 + \frac{2}{\mu}\right), \tag{2}$$

where μ is the static amplification factor of the input tube, then the fundamental component is completely compensated and the parasitic signal is transformed to a spectrum of even harmonics, as can be seen from the oscillogram in Fig. 1.

TABLE 1

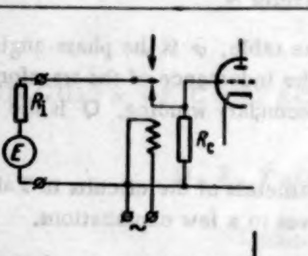
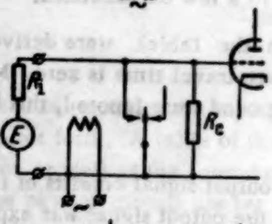
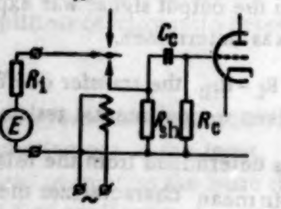
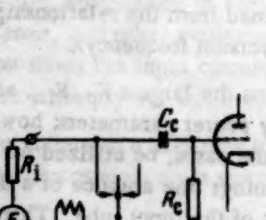
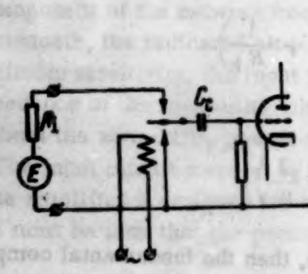
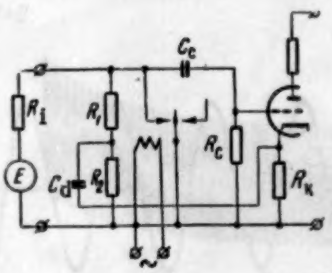
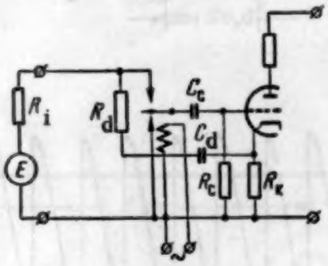
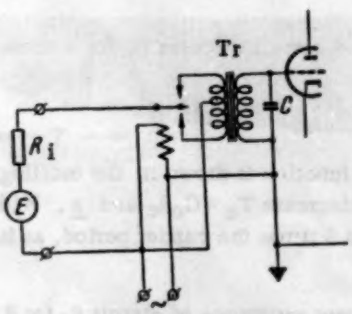
Circuit No.	Circuit	Basic parameters
1		$K_p = \frac{\sqrt{2}}{\pi}\frac{R_c}{R_c + R_i}$, $R_{in} = R_c$, $R_{in\,mean} = 2R_c$, $\alpha = \frac{2\sqrt{2}}{\pi}$.
2		$K_p = \frac{\sqrt{2}}{\pi}\frac{R_c}{R_c + R_i}$, $R_{in} = R_c$, $R_{in\,mean} = 0$, $\alpha = \frac{2\sqrt{2}}{\pi}$.
3		$K_p = \frac{\sqrt{2}}{\pi}\frac{2R_{sh}R_c}{R_{sh}(R_{sh}+2R_c)+2R_i(R_{sh}+R_c)}$, $R_{in} = R_{sh}\frac{R_{sh}+2R_c}{2(R_{sh}+R_c)}$, $R_{in\,mean} = R_{sh}\frac{R_{sh}+2R_c}{R_{sh}+R_c}$, $\alpha \approx 0.9$.
4		$K_p = \frac{\sqrt{2}}{\pi}\frac{2R_c}{2R_c + R_i}$, $R_{in} = 2R_c$, $R_{in\,mean} = 0$, $\alpha = \frac{2\sqrt{2}}{\pi}$.
5		$K_p = \frac{\sqrt{2}}{\pi}\frac{R_c}{R_c + R_i}$, $R_{in} = R_c$, $R_{in\,mean} = 2R_c$, $\alpha \approx \frac{1}{\sqrt{2}\,\lvert\cos\varphi\rvert}$.
6		$K_p = \frac{\sqrt{2}}{\pi}\frac{2R_c}{2R_c + R_i}$, $R_{in} = 2R_c$, $R_{in\,mean} = 4R_c$, $\alpha = \frac{2\sqrt{2}}{\pi}$.

TABLE 1 (Continued)

Circuit No.	Circuit	Basic Parameters
7		$K_p = \frac{\sqrt{2}}{\pi}\frac{2R_c(R_1+R_2)}{(R_1+R_2)(R_i+2R_c)+2R_iR_c}$, $R_{\text{in}} = \frac{2R_c(R_1+R_2)}{R_1+R_2+2R_c}$, $R_{\text{in mean}} = 0$, $\alpha = \infty$.
8		$K_p = \frac{\sqrt{2}}{\pi}\frac{2R_c}{2R_c+R_i}$, $R_{\text{in}} = 2R_c$, $R_{\text{in mean}} = 4R_c$, $\alpha = \infty$.
9		$K_p = \frac{2\sqrt{2}}{\pi}\frac{R_{\text{in}}}{R_i+R_i}$, $R_i = \frac{L_2}{RCn^2} = \frac{Q}{2\pi f_0 C n^2}$, $R_{\text{in mean}} = R_i$, $\alpha = \infty$.

The conversion factor for the useful signal in circuits 7 and 8 is the same as in circuit 6. The input resistance of circuits 8 and 6 is also equal. The input resistance of circuit 7 is considerably lower and thereby limits its applications.

We can conclude from the comparison of the transformerless circuits that circuits 6 and 8 are the most suitable for application in autocompensator amplifiers. Let us establish other properties of these circuits. A more accurate value of K_p for circuit 6 can be determined from the formula

$$K_p = \frac{\sqrt{2}}{\pi}\frac{2R_c}{2R_c+R_i}\sqrt{1-\frac{1}{4f_0C_cR_c}\frac{R_i+2R_c}{R_i+R_c}}. \tag{3}$$

The phase shift of the fundamental frequency relative to the phase angle of the contact switching is determined from the expression

$$\tan\varphi = \frac{1}{2\omega C_cR_c\left(\frac{R_c+R_i}{2R_c+R_i}-\frac{1}{8f_0C_cR_c}\right)}. \tag{4}$$

Since the condition of constant phase angle [1] is met in autocompensator amplifiers, the dynamic properties of circuit 6, relative to the envelope, can be characterized by the transfer function which, in operator form (for $2f_0C_cR_c \gg 1$) has the form:

$$f(p) = \frac{\sqrt{2}}{\pi}\,\frac{R_c}{R_c + R_i}\left(1 + \frac{\pi}{\omega C_c R_c}\right)\left[1 + \frac{a-1}{1 + pC_cR_c a}\right], \tag{5}$$

where p is the differentiation operator and $a = \dfrac{2(R_c + R_i)}{(2R_c + R_i)}$.

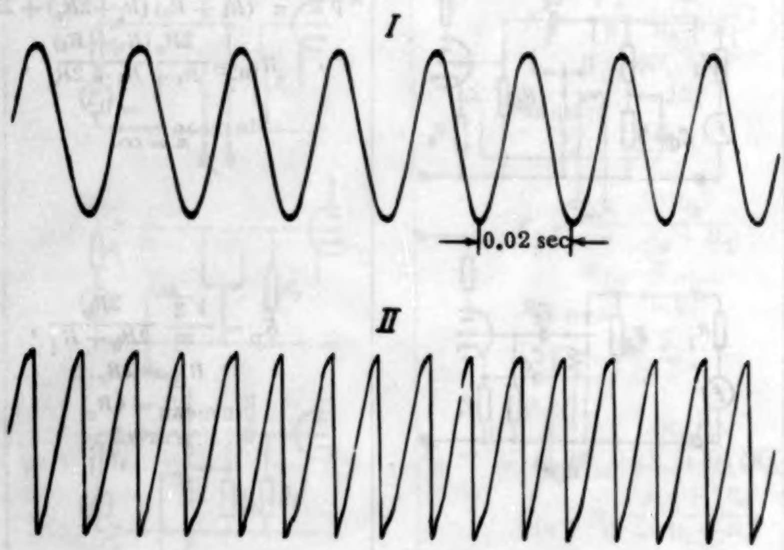


Fig. 1. Shape of the output voltage of circuit 8 (curve II) for a sinusoidal input (curve I).

The transient response of the circuit to a unit step function is shown in the oscillograms in Fig. 2. Obviously, to decrease the transient time it is necessary to decrease $T_c = C_cR_c$ and a. With proper choice of parameters the transient time can be reduced to a value 2 to 3 times the carrier period, as is apparent from the oscillogram in Fig. 2,c.

It should be noted that an accurate value for the input resistance of circuit 6, for $R_c > 10^7$ ohms, cannot be obtained from the formula given in the table because, under these conditions, the conductivity associated with the overcharging of the contact capacitances and the capacitance of the reed to ground [1] has an appreciable value.

The dynamic properties of circuit 8 are somewhat poorer than those of circuit 6, especially for large values of R_i (Fig. 3, a), which is involved in the overcharging of capacitor C_d. Therefore, it is expedient to use circuit 6 when a high input resistance is necessary.

To reduce the transient time, the following process for calculating circuit 8 is recommended:

1. Choose the value of R_k for normal self-bias conditions.

2. Knowing the required input resistance R_{in}, choose $R_c = \dfrac{R_{in}}{2}$.

3. Knowing R_k and R_i, it is necessary to choose a time constant $T_t = C_d\left[R_i + 2R_k\left(1 + \dfrac{1}{\mu}\right)\right]$, whence C_d is determined (T_t determines the transient time).

4. From the condition of equal phase shifts in the grid and cathode circuits, C_c is determined:

$$C_c \approx C_d\,\frac{2R_k}{R_c} + \frac{1}{4f_0R_c}. \tag{6}$$

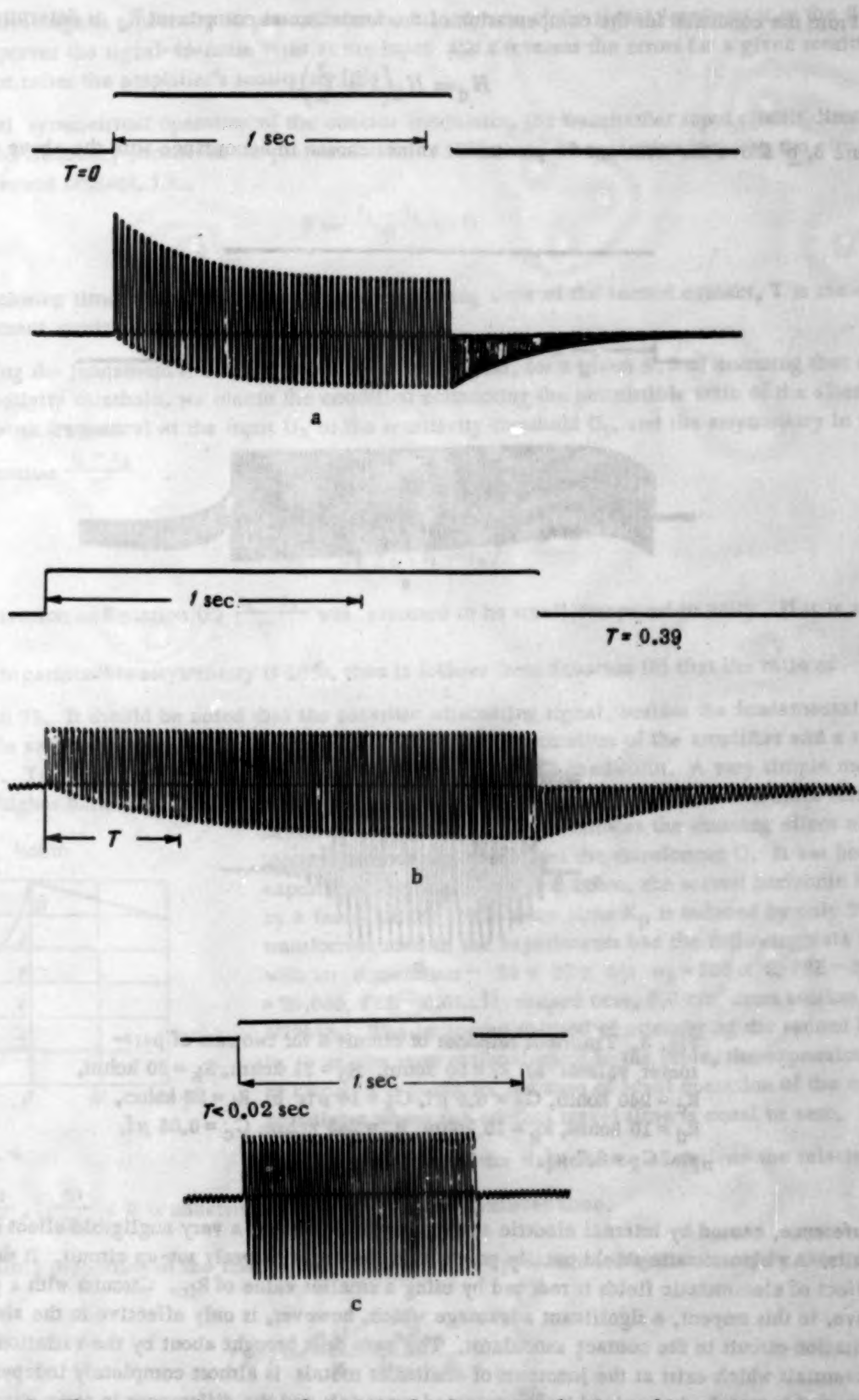


Fig. 2. The transient response of circuit 6 for three sets of values of its parameters: a) $R_l = 0$, $T_c = 0.2$ sec; b) $R_l = 1.5 \times 10^6$ ohms, $T_c = 0.2$ sec; c) $R_l = 1.5 \times 10^6$ ohms, $T_c = 0.02$ sec.

5. From the condition for the compensation of the fundamental component, R_d is determined:

$$R_d = R_k\left(1 + \frac{2}{\mu}\right).$$

Figure 3, b shows the transient for parameter values chosen in accordance with the above suggested method.

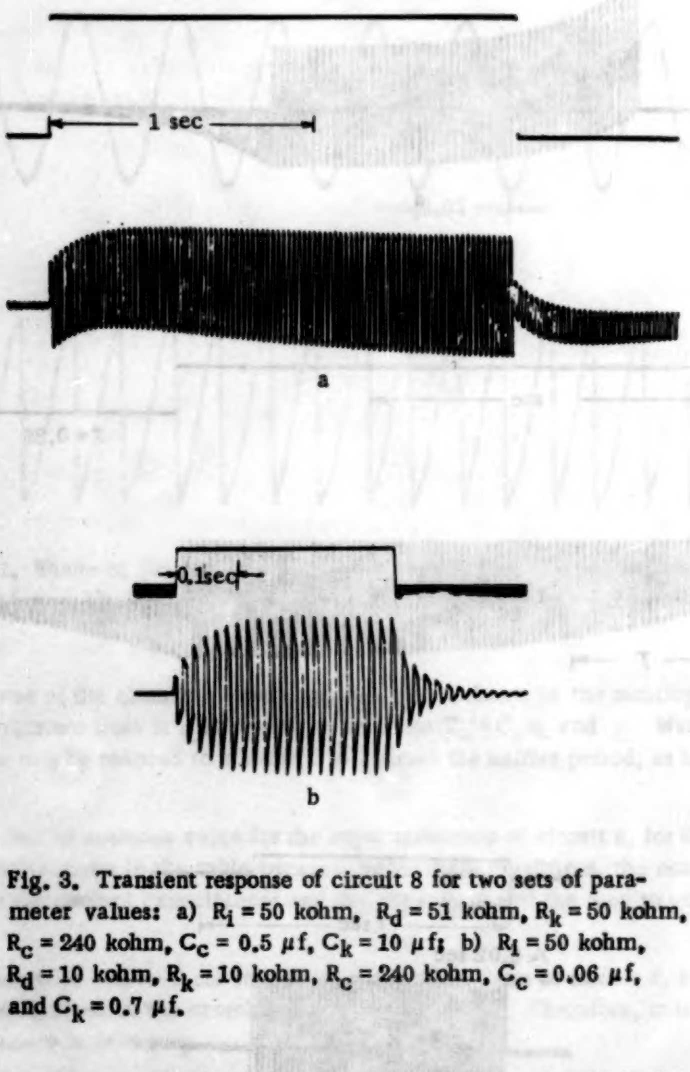


Fig. 3. Transient response of circuit 8 for two sets of parameter values: a) $R_i = 50$ kohm, $R_d = 51$ kohm, $R_k = 50$ kohm, $R_c = 240$ kohm, $C_c = 0.5\ \mu f$, $C_k = 10\ \mu f$; b) $R_i = 50$ kohm, $R_d = 10$ kohm, $R_k = 10$ kohm, $R_c = 240$ kohm, $C_c = 0.06\ \mu f$, and $C_k = 0.7\ \mu f$.

Interference, caused by internal electric and magnetic fields, has a very negligible effect on transformerless input circuits. An electrostatic shield usually proves sufficient in a properly set-up circuit. It should be noted that the effect of electrostatic fields is reduced by using a smaller value of R_{in}. Circuits with a grounded moving contact have, in this respect, a significant advantage which, however, is only effective in the absence of screening of the excitation circuit in the contact modulator. The zero drift brought about by the variation of the thermoelectric potentials which exist at the junctions of dissimilar metals is almost completely independent of the circuit chosen, and is mainly determined by the selected materials and the differences in temperature between various points in the input circuit.

Let us consider the converter with transformer input (circuit 9). This circuit has found wide application, especially in autocompensator amplifiers. The main advantages of this circuit are: 1) isolation of the input is simple [1]; 2) a large value of the ac selectivity coefficient can be easily achieved; 3) a large value of K_p is easily attainable.

The last advantage is especially important because an increase of the signal level, prior to the first amplifier stage, improves the signal-to-noise ratio at the input and decreases the errors for a given sensitivity or, for the same error, raises the amplifier's sensitivity [2].

Under ideal symmetrical operation of the contact modulator, the transformer input circuit discriminates against conversion-frequency parasitics. In practice, a finite difference always exists between the closing time of the first and second contact, i.e.,

$$\beta = \frac{t_1 - t_2}{T} \neq 0, \tag{7}$$

where t_1 is the closing time of the first contact, t_2 is the closing time of the second contact, T is the operating period of the contact modulator.

Determining the fundamental component of the output signal, for a given β, and assuming that its value is below the sensitivity threshold, we obtain the condition connecting the permissible ratio of the alternating component (network frequency) at the input U_a to the sensitivity threshold U_t, and the assymmetry in the contact modulator's operation $\frac{t_1 - t_2}{T}$.

$$\frac{u_a}{u_t} \leqslant \frac{3}{\sqrt{2}\pi}\left(\frac{T}{t_1 - t_2}\right)^3. \tag{8}$$

In the derivation of Equation (8) $\frac{t_1 - t_2}{T}$ was assumed to be small compared to unity. If it is assumed that the maximum permissible assymmetry is 10%, then it follows from Equation (8) that the ratio of $\frac{U_a}{U_t}$ can be no larger than 75. It should be noted that the parasitic alternating signal, besides the fundamental, results in an appreciable value of the second harmonic which may lead to saturation of the amplifier and a reduction in its sensitivity. To avoid this it is necessary to reduce the amplifier's bandwidth. A very simple means of suppressing the higher harmonics is the use of a tuned transformer and the introduction of an extra resistor R_1, in series with the input, which reduces the shunting effect of the source's input resistance and thus raises the transformer Q. It has been established experimentally that, for R_1 = 8 kohm, the second harmonic is attenuated by a factor of 10; at the same time K_p is reduced by only 30%. The transformer used in the experiments had the following data associated with it: dimensions – 24 × 32 × 44; w_1 = 800 × 2, PEL – 0.1; w_2 = = 20,000, PEL – 0.05; П-shaped core, 0.8 cm^2 cross section, permalloy N79M5. The indicated method of attenuating the second harmonic is, to us, the most rational one. In the table, the expression for R_{in} in circuit 9 is given for the case of ideal operation of the contact-modulator where the contact travel time is equal to zero.

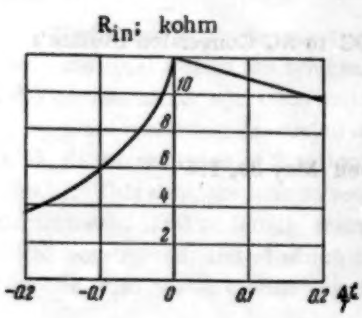


Fig. 4

Figure 4 shows the dependence of R_{in} on the relative, contact-travel time $\frac{\Delta t}{T}$. $\frac{\Delta t}{T}$ < 0 is understood to be the contact flashover time.

The dynamic properties of the transformer circuit can be approximated quite closely by the transfer function

$$f(p) = \frac{2\sqrt{2}}{\pi} \frac{nR_{in}}{R_{in} + R_1} \frac{1}{1 + \frac{Q}{\pi f_0} p}, \tag{9}$$

where Q is the transformer quality factor, f_0 is the conversion frequency, n is the transformation factor, p is the differentiating operator.

The following shortcomings of circuit 9 should be noted:

1) appreciable sensitivity to internal, alternating, magnetic fields which necessitates the use of specially constructed transformers and thorough screening [1];

2) difficulty of achieving a high input impedance;

3) relatively high cost.

In concluding, the following recommendations on the use of the described circuits with contact converters can be made :

1. In those cases where isolation of the inputs is not required and the permissible sensitivity threshold is 10 microvolts, the most rational choice is the transformerless circuit 8.

2. If an isolated input is required and a sensitivity threshold of less than 10 microvolts is necessary, the advantages of the transformer circuit are obvious (circuit 9).

3. The application of circuit 6 is expedient in those cases where a very large input impedance with high dynamic characteristics is required, as in the measurement of small currents.

4. The application of the other tabulated circuits in autocompensator amplifiers is, in general, inefficient.

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A PUNCHED CARD METHOD FOR SYNTHESIZING SEQUENTIAL RELAY SYSTEMS

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A method is presented for implementing a vector-algebraic method of synthesizing sequential systems of two-position relays by means of special punched cards. This method is applicable for the synthesis of both autonomous and nonautonomous relay systems.

1. Introduction

For facilitating and speeding the process of synthesizing relay circuits, both relay contact and vacuum tube circuits, a number of punched card methods have been proposed, but in all the works [1-6] presenting these methods what was discussed was only the minimization of the expression of the Boolean functions describing the structure of the relay circuit to be synthesized. These methods are quite effective since they allow a significant portion of the work and time expended on simplifying the circuit to be synthesized to be saved. They are particularly effective in the synthesis of one-cycle or, as they are still called, combinatory circuits, since the simplification process is the most difficult step in the synthesis of such circuits.

These card methods may also be used for the synthesis of sequential relay circuits. However, simplification is only the final step in the synthesis of sequential relay circuits. This step is overshadowed by the determination of the conditions for operation and release of each of the relays in the circuit to be synthesized. The determination of these conditions is carried out from data, given in some manner (for example, by means of a switching table), on the sequence of functioning of all the relays in the desired system for given sequences of external stimuli (signals). This step, specific to sequential relay systems, is completely independent of the concrete elements (electromechanical or ferrite relays, vacuum tube or crystal diodes, etc.) from which the circuit to be synthesized will be constructed, and of which the equivalent relay circuits of the system will be synthesized. Thus, we replace the term "relay circuit" by the more general term "relay system," introduced by the author in [7].

A vector-algebraic method [8-10] is proposed for the synthesis of sequential relay systems. It is always possible, by using this method, to obtain Boolean functions formulating the necessary and sufficient conditions for the functioning of each relay in the system to be synthesized, given the sequence of functioning of all relays in the system to be synthesized and the sequence of external stimuli to this system. The computation of these functions is very simple and does not require much time if the number of independent variables (external signals) and the number of relays in the system to be synthesized are small. However, with an increase in the number of independent variables and relays in the system to be synthesized, there is a very rapid increase in the length of the sequences of external signals and of the sequences of relay states which are to be synthesized. Because of this, the computations necessary for the synthesis of sequential relay systems with an increase of independent signals and relays rapidly become ever more extensive and burdensome, with the result that the probability of error in the synthesis also increases. The requirement arises of facilitating and speeding, in some measure, the necessary computations by means of mechanization of at least the simplest of them.

This paper presents one of the simplest methods for mechanizing the vector-algebraic method of synthesizing sequential relay systems. This method is based on the use of special cards, which gives it its name, a punched card method of synthesizing sequential relay systems. For the exposition of this method we shall assume that the reader is familiar with the vector-algebraic methods of synthesizing sequential relay systems. Hence, we shall

give here only the minimum details of this method as required for the understanding of the method, presented here, of using special cards for the mechanization of the vector-algebraic method of synthesis.

2. Short Exposition of the Vector-Algebraic Method of Synthesizing Sequential Relay Systems

At any moment of time t, the state of the n-relay system $Y_1, \ldots, Y_n$ can be unambiguously defined by giving two vectors: the vector of the independent parameters $\mathbf{x}(t) = [x_1(t), \ldots, x_i(t), t, \ldots, x_m(t)]$, controlling the given relay system, and the vector of the dependent parameters $\mathbf{y}(t) = [y_1(t), \ldots, y_k(t), \ldots, y_n(t)]$, controlled by means of the relays of the given system. When the relays of the given system are electromechanical, the variables $x_1, \ldots, x_i, \ldots, x_m$ are contacts closed by relays or buttons $X_1, \ldots, X_i, \ldots, X_m$, external to the given relay system, and the variables $y_1, \ldots, y_k, \ldots, y_n$ are contacts closed by operation of the corresponding relays $Y_1, \ldots, Y_k, \ldots, Y_n$ of the given system.

If all the relays in the given system have identical lags in operating and releasing, and if all the controlling and controlled parameters are varied in virtual synchronism, then the process of changing state of this system can be described by the following vector equation:

$$\mathbf{y}(t+\tau) = \mathbf{f}(\mathbf{x}(t), \mathbf{y}(t)), \tag{1}$$

where $\mathbf{f}$ is a single-valued vector function.

Determining the process $\mathbf{y}(t)$ by giving the function $\mathbf{f}$ and the process $\mathbf{x}(t)$ is called analysis, while computing the function $\mathbf{f}(\mathbf{x}, \mathbf{y})$ by giving the processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is called the synthesis of the system described by Equation (1).

Under the assumptions made above with respect to relay functioning in the given system, it is possible to replace the processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ by the sequences $\mathbf{x}(j)$ and $\mathbf{y}(j)$, where $j = 0, 1, 2, \ldots$. Then Equation (1) is replaced by the following equation:

$$\mathbf{y}(j+1) = \mathbf{f}(\mathbf{x}(j), \mathbf{y}(j)), \tag{2}$$

where the natural* variable j ($j = 0, 1, 2, \ldots$) will be called the cycle number of the sequential (multicycle) relay system described by Equation (2).

A relay system is termed autonomous if each controlled parameter in it is controlled only by relays lying in the same system. Such a system is described by an equation of the form

$$\mathbf{y}(j+1) = \mathbf{f}(\mathbf{y}(j)). \tag{3}$$

A set of several sequences $\mathbf{y}(j)$ is called complete if among the terms of this sequence all 2^n values $\mathbf{y}_\beta$ ($\beta = 0, 1, \ldots, 2^n - 1$) of the variable vector $\mathbf{y}$ occur.

It follows from the single-valuedness of the function $\mathbf{f}$ that for the synthesis of autonomous systems only such complete sets of sequences $\mathbf{y}(j)$ are suitable for which each pair of sequences $\mathbf{y}^{(r)}(j)$ and $\mathbf{y}^{(s)}(j)$ satisfies the condition

$$\text{if} \quad \mathbf{y}^{(r)}(j^{(r)}) = \mathbf{y}^{(s)}(j^{(s)}), \text{then } \mathbf{y}^{(r)}(j^{(r)}+1) = \mathbf{y}^{(s)}(j^{(s)}+1), \tag{A}$$

where r and s independently run through the values $1, 2, \ldots, N$ (N is the number of all the sequences $\mathbf{y}(j)$ in the given complete set).

In contradistinction to autonomous systems there are nonautonomous relay systems. In particular, if a relay system does not contain an autonomous subsystem it is called completely nonautonomous, or a single-cycle system.

An algorithm for synthesizing autonomous systems of two-position relays from a given complete set of sequences $\mathbf{y}(j)$, satisfying Condition (A), is given, as was shown in [8], by the formula

* By a natural variable we mean a variable which runs through the series of natural numbers: 0, 1, 2, 3, ...

$$f(y)=\sum_{r=1}^{N}\sum_{j=0}^{N_r-1} p_{\beta_j^{(r)}}(y)\, y^{(r)}(j+1), \tag{4}$$

where Σ is the sign for the Boolean sum, N is the total number of sequences in the given complete set, N_r is the number of states in the r-th sequence and $p_\beta(y)$ are constituent units computed, as shown in [9], from the vector components by the formula

$$p_\beta(y)=\prod_{k=1}^{n}(y_{\beta,k}\oplus y_k\oplus 1), \tag{5}$$

where y_k and $y_{\beta,k}$ are, respectively, the components of the variable vector y and its values y_β, and $\oplus$ is the sign for mod 2 addition.

The algorithm for synthesizing nonautonomous systems of two-position relays, i.e., relay systems described by Equation (2), is given, as shown in [10], by the formula

$$f(x,y)=\sum_{r=1}^{N}\sum_{j=0}^{N_r-1} p_{\alpha_j^{(r)}}(x)\, p_{\beta_j^{(r)}}(y)\, y^{(r)}(j+1), \tag{6}$$

where the $p_\alpha(x)$ are constituent units, computed from the components of the variable vector x and its values, x_α by a formula analogous to Formula (5). Formula (6) is a generalization of Formula (4).

Sequences x (j) and y (j) for which the function f(x, y) may be computed from Formula (6) must satisfy the condition

$$\text{if} \quad x^{(r)}(j^{(r)})=x^{(s)}(j^{(s)}) \text{ and } y^{(r)}(j^{(r)})=y^{(s)}(j^{(s)}), \tag{B}$$

$$\text{then} \quad y^{(r)}(j^{(r)}+1)=y^{(s)}(j^{(s)}+1),$$

where r and s independently run through the values 0, 1, 2, . . . , N (N is the number of pairs of sequences x (j) and y(j) in the given complete set). Condition (B) is a generalization of Condition (A).

If the given sequence does not initially satisfy Conditions (A) and (B) then, in many cases, it may be extended so that these conditions are satisfied, this extension being carried out perhaps by the method described in [9].

3. The Use of Special Cards for the Synthesis of Autonomous Relay Systems

The values y_β of the variable vector y, characterizing the state of the relay system $Y_1, \ldots, Y_k, \ldots, Y_n$, may be numbered in different ways. As in earlier works [10, 8], we will number them by the numbers β, obtained from the components $y_{\beta,k}=b_k$ of the vector y_β, by the rule expressed by the formula

$$\beta=\sum_{k=1}^{n} b_k 2^{k-1}. \tag{7}$$

Conversely, by using this rule it is easy, being given the number β of an arbitrary value y_β of the variable vector y, to write all the components of this value. For this it suffices to write the given number β in the binary system, $\beta=b_nb_{n-1}\ldots b_k\ldots b_2b_1$. The binary digits of this number, read from right to left, will be the components of the desired value $y_\beta=(b_1, b_2, \ldots, b_k, \ldots, b_{n-1}, b_n)$. Precisely by this method the values of the vector y are numbered in the table.

Using (5) for each value y_β of the vector y, we obtain the corresponding constituent units $p_\beta(y)$. However, the expressions for the constituents $p_\beta(y)$ are easily obtained from the corresponding values y_β by the following rule: replace each component b_k of the vector y by the factor y_k if $b_k=1$, and by the factor y'_k (i.e., the inverse of the value of y_k) if $b_k=0$. The values of 32 constituents $p_\beta(y)$ are given in the third column of the Table.

Since the components b_k of the vector y_β can take only two values, 0 and 1, there is no difficulty in preparing cards simulating the vectors y_β. For this, it suffices to define the places (positions) on card number β, simulating the vector y_β, assigned to the components b_k of vector y_β, and then to make notches or perforations only in those positions corresponding to values of $b_k = 1$. The positions $\underline{k}$, corresponding to the components b_k of vector y_β, are most expeditiously ordered linearly, placing them, for example, along one of the sides of the card in increasing order of $\underline{k}$.

TABLE

β	$y_\beta = [b_1, b_2, b_3, b_4, b_5]$	$p_\beta(y)$	β	$y_\beta = [b_1, b_2, b_3, b_4, b_5]$	$p_\beta(y)$
0	0 0 0 0 0	$y_1'\ y_2'\ y_3'\ y_4'\ y_5'$	16	0 0 0 0 1	$y_1'\ y_2'\ y_3'\ y_4'\ y_5$
1	1 0 0 0 0	$y_1\ y_2'\ y_3'\ y_4'\ y_5'$	17	1 0 0 0 1	$y_1\ y_2'\ y_3'\ y_4'\ y_5$
2	0 1 0 0 0	$y_1'\ y_2\ y_3'\ y_4'\ y_5'$	18	0 1 0 0 1	$y_1'\ y_2\ y_3'\ y_4'\ y_5$
3	1 1 0 0 0	$y_1\ y_2\ y_3'\ y_4'\ y_5'$	19	1 1 0 0 1	$y_1\ y_2\ y_3'\ y_4'\ y_5$
4	0 0 1 0 0	$y_1'\ y_2'\ y_3\ y_4'\ y_5'$	20	0 0 1 0 1	$y_1'\ y_2'\ y_3\ y_4'\ y_5$
5	1 0 1 0 0	$y_1\ y_2'\ y_3\ y_4'\ y_5'$	21	1 0 1 0 1	$y_1\ y_2'\ y_3\ y_4'\ y_5$
6	0 1 1 0 0	$y_1'\ y_2\ y_3\ y_4'\ y_5'$	22	0 1 1 0 1	$y_1'\ y_2\ y_3\ y_4'\ y_5$
7	1 1 1 0 0	$y_1\ y_2\ y_3\ y_4'\ y_5'$	23	1 1 1 0 1	$y_1'\ y_2\ y_3\ y_4'\ y_5$
8	0 0 0 1 0	$y_1'\ y_2'\ y_3'\ y_4\ y_5'$	24	0 0 0 1 1	$y_1'\ y_2'\ y_3'\ y_4\ y_5$
9	1 0 0 1 0	$y_1\ y_2'\ y_3'\ y_4\ y_5'$	25	1 0 0 1 1	$y_1\ y_2'\ y_3'\ y_4\ y_5$
10	0 1 0 1 0	$y_1'\ y_2\ y_3'\ y_4\ y_5'$	26	0 1 0 1 1	$y_1'\ y_2\ y_3'\ y_4\ y_5$
11	1 1 0 1 0	$y_1\ y_2\ y_3'\ y_4\ y_5'$	27	1 1 0 1 1	$y_1\ y_2\ y_3'\ y_4\ y_5$
12	0 0 1 1 0	$y_1'\ y_2'\ y_3\ y_4\ y_5'$	28	0 0 1 1 1	$y_1'\ y_2'\ y_3\ y_4\ y_5$
13	1 0 1 1 0	$y_1\ y_2'\ y_3\ y_4\ y_5'$	29	1 0 1 1 1	$y_1\ y_2'\ y_3\ y_4\ y_5$
14	0 1 1 1 0	$y_1'\ y_2\ y_3\ y_4\ y_5'$	30	0 1 1 1 1	$y_1'\ y_2\ y_3\ y_4\ y_5$
15	1 1 1 1 0	$y_1\ y_2\ y_3\ y_4\ y_3'$	31	1 1 1 1 1	$y_1\ y_2\ y_3\ y_4\ y_5$

Fig. 1. Cards 0, 1, 2, 3, . . . , 16, . . . , 30, 31, for the synthesis of autonomous relay systems.

Since there is a one-to-one correspondence between the values y_β of the vector y and the constituents $p_\beta(y)$ having the same number β, simulation of the vector y can be considered as simultaneously simulating the constituents $p_\beta(y)$. With this, a notch or perforation in position $\underline{k}$ of card number β corresponds to the factor y_k in the constituent $p_\beta(y)$, and the absence of a notch or perforation in this position corresponds to the factor y_k' of the same constituent. Now, if opposite each position $\underline{k}$ of card number β is written the designation p_β of the constituent $p_\beta(y)$, corresponding to the given card, then the cards thus obtained may be used for the synthesis of autonomous relay systems in accordance with Formula (4). We immediately add that, technically, it is more

effective to place, not the constituent designations $p_\beta(y)$ on the cards, but only their numbers β, with this proviso, that the numbers be large enough (but not larger than the notches or perforations) so that when one card is laid on another, the number of the lower card might be read through the perforations of the upper card.

Figure 1 shows the cards, prepared by this method, for the values $\beta = 0, 1, 2, 3, \ldots, 16, \ldots, 30, 31$.

As is obvious from Fig. 1, card 0 (card number 0) has no notches, card 1 has a notch in the first position, card 2 in the second position, card 3 in the first and second positions, card 16 in the fifth position, card 30 in positions two, three, four, and five, and card 31 in all five positions.

As is well known [7, 8], the process of changing the state of an autonomous system may be given algebraically either by the sequence of values y_{β_j} of the vector $y(j)$, describing this system, or by the sequence of constituents $p_{\beta_j}(y)$, corresponding to these values of the vector y. Therefore, the sequence in which the cards, β_j are placed can simulate the process occurring in the autonomous relay system, whether we consider these cards as simulating the values y_{β_j} of the vector $y(j)$, or as simulating the corresponding constituents $p_{\beta_j}(y)$. With this, the left sides of the cards β, containing the notches or perforations, are best considered as models of the values y_β of vector y, and the right sides of the same cards, filled with the numbers β, as the bearers of the constituents $p_\beta(y)$, i.e., the number β is considered as a shortened notation of the constituent $p_\beta(y)$ itself. Then, if card β_j is laid on card β_j' so that the left (i.e., perforated) part lies on the right (i.e., the part bearing the number β_j) part of the first card, then through the notches of the second card will be seen the number β_j of the first card. If we consider the left side of card β_j' as the model of vector $y_{\beta_j'}$, then the left side of card β_j', lying on the right side of card β_j can be considered as modelling the product of the vector $y_{\beta_j'}$ by the constituent $p_{\beta_j}(y)$, i.e., as modelling the expression $p_{\beta_j}(y)\, y_{\beta_j'}$. If the positions of the cards are reversed, we obtain the model of the other expression, $p_{\beta_j'}(y)\, y_{\beta_j}$.

For example, let card 7 be placed on card 3 in the manner described. This gives a model of the expression $p_3 y_7 = p_3\,[1, 1, 1, 0, 0] = [p_3, p_3, p_3, 0, 0]$, shown in Fig. 2.

Fig. 2

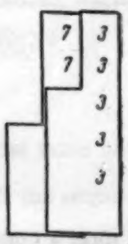

Fig. 3

Through the notches on upper card 7 are seen the numbers 3 on positions one, two, and three, while positions 4 and 5 of card 3 are covered by the nonperforated portion of card 7. This denotes that card 7, lying on card 3 as shown in Fig. 2, models the vector $[p_3, p_3, p_3, 0, 0]$, i.e., the expression $p_3 y_7$. If cards 3 and 7 are interchanged, we obtain the model, shown in Fig. 3, of the expression $p_7 y_3 = p_7\,[1, 1, 0, 0, 0] = [p_7, p_7, 0, 0, 0]$. As shown in Fig. 3, through the notches of upper card 3 are seen the numbers 7 on positions one and two, and on the remaining positions the numbers 7 are covered over by the nonperforated portion of card 3. This denotes that placing card 3 on card 7 gives a model of the vector $[p_7, p_7, 0, 0, 0]$, i.e., of the expression $p_7 y_3$.

Thus, the result obtained by placing one card on another card depends on the order or sequence in which the placement is effected.

Let there be given a certain sequence $y(j) = y_{\beta_j}$ of values y_{β_j} of the vector y characterizing the states of some autonomous relay system. This sequence will terminate, as known from [7, 9], in some periodic process or, in particular, in some stable state, i.e., in the general case the sequence will have the form

$$y_{\beta_0}, y_{\beta_1}, \ldots, y_{\beta_{j_*-1}}, (y_{\beta_{j_*}}, \ldots, y_{\beta_{j_*+m-1}}), \tag{8}$$

where the sequence in brackets is a periodic sequence.

If cards β_j are chosen in the sequence corresponding to Sequence (8), i.e., in the sequence

$$\beta_0, \beta_1, \ldots, \beta_{j_*-1}, \beta_{j_*}, \ldots, \beta_{j_*+m-1}, \beta_{j_*}, \tag{9}$$

and laid one on the other such that the right (i.e., the constituent number-bearing) part of each previous card β_j will be overlapped by the left (i.e., perforated) part of the following card in the sequence β_{j+1}, then the interlacing of cards β_j and β_{j+1} will model the expression $p_{\beta_j}(y)y_{\beta_{j+1}} = p_{\beta_j}(y)y\ (j+1)$.

If the given sequence is just one of several in a given complete set of sequences then, using a superscript index $\underline{r}$ for the given sequence, we have that the card β_{j+1}, lying, as described above, on card β_j, models the expression $p_{\beta_j^{(r)}}(y)\,y^{(r)}(j+1)$.

The set or, more accurately, the union of all such expressions, obtained from all $N_r + 1$ cards β_j, lying one atop the other in the fashion described above, in the Sequence (9) is the Boolean sum

$$\sum_{j=0}^{N_r-1} p_{\beta_j^{(r)}}(y)\,y^{(r)}(j+1), \tag{10}$$

contained in Formula (4).

By carrying out such an operation for each of the sequences $y^{(r)}(j)$, and finding the union of all N Boolean sums (10) thus obtained, we get the right member of Formula (4) and, thus, the sought for function itself, $f(y)$. The components $f_k(y)$ of this function are the conditions for operating the corresponding relays Y_k of the autonomous system to be synthesized.

To illustrate the punched card method of synthesizing autonomous relay systems, we consider the following example.

Let it be required to implement the autonomous relay system in which the following processes would occur [7, 8]:

$$P(y_0) = (y_0,\ y_1,\ y_3,\ y_7,\ y_{15},\ y_{11},\ y_9,\ y_8),$$
$$P(y_2) = y_2,\ y_5,\ (y_{11},\ y_9, \ldots,\ y_{15}),$$
$$P(y_4) = y_4,\ (y_9,\ y_8, \ldots,\ y_{11}),$$
$$P(y_6) = y_6,\ y_{13},\ y_{10},\ (y_1,\ y_3, \ldots,\ y_0),$$
$$P(y_{12}) = y_{12},\ (y_8,\ y_0, \ldots,\ y_9),$$
$$P(y_{14}) = y_{14},\ (y_9,\ y_8, \ldots,\ y_{11}).$$

All these sequences converge to one and the same periodic sequence $P(y_0)$ and form a complete set of sequences, since they contain all 16 possible states of the four relays Y_1, Y_2, Y_3, and Y_4.

To find the function $f(y)$ describing the desired autonomous system, we first find the functions $f^{(r)}(y)$ $(r = 1, 2, \ldots, 6)$, in each of which there occurs just one of the given six processes: $P(y_0)$, $P(y_2)$, ..., $P(y_{14})$. The desired function $f(y)$ will obviously be the Boolean sum of the six functions $f^{(r)}(y)$.

We first find the autonomous system, the changes of state of which would be described by the sequence $P(y_0)$, where we require that this process be maximally stable, i.e., that from all the initial states which are not included in the given periodic process, the system will immediately transfer to state y_0, the initial phase of the given periodic process $P(y_0)$.

By stacking the cards β_j according to the method described above, in accordance with the values y_{β_j} entering into the given sequence $P(y_0)$, we obtain the disposition of cards 0, 1, 3, 7, 15, 11, 9, 8, 0, shown in Fig. 4. By considering, in accordance with the conventions we have adopted, that the left side of card β_j models the vector y_{β_j} and the right side of the same card models the constituent p_{β_j}, we see from Fig. 4 that

$$p_0y_1 = [p_0,\ 0,\ 0,\ 0], \qquad p_1y_3 = [p_1,\ p_1,\ 0,\ 0],$$

$$p_3y_7 = [p_3, p_3, p_3, 0], \qquad p_7y_{15} = [p_7, p_7, p_7, p_7],$$
$$p_{15}y_{11} = [p_{15}, p_{15}, 0, p_{15}], \qquad p_{11}y_9 = [p_{11}, 0, 0, p_{11}],$$
$$p_9y_8 = [0, 0, 0, p_9], \qquad p_8y_0 = [0, 0, 0, 0] = \mathbf{0}.$$

Taking the union of all these expressions, i.e., finding their Boolean sum, we find the desired function:

$$\mathbf{f}^{(1)}(\mathbf{y}) = p_0y_1 + p_1y_3 + p_3y_7 + p_7y_{15} + p_{15}y_{11} + p_{11}y_9 + p_9y_8 + p_8y_0,$$

where "+" is the sign for Boolean addition. We note that the last term here is zero, since $\mathbf{y}_0 = \mathbf{0}$, and hence it might have been omitted.

If we now replace the terms found above by their values, and carry out the addition according to the rules for vector addition, we obtain the components $f_k^{(1)}(\mathbf{y})$ of the vector function $\mathbf{f}^{(1)}(\mathbf{y})$:

$$f_1^{(1)}(\mathbf{y}) = p_0 + p_1 + p_3 + p_7 + p_{11} + p_{15}, \qquad f_2^{(1)}(\mathbf{y}) = p_1 + p_3 + p_7 + p_{15},$$

$$f_3^{(1)}(\mathbf{y}) = p_3 + p_7, \qquad f_4^{(1)}(\mathbf{y}) = p_7 + p_9 + p_{11} + p_{15}.$$

By comparing the numbers of the constituents in the expression for the k-th component $f_k^{(1)}(\mathbf{y})$ of the function $\mathbf{f}^{(1)}(\mathbf{y})$ with the digits which are visible in the k-th row of the cards distributed as in Fig. 4, we see that the digits which can be seen through the notches in the k-th row, distributed in the manner described above, are the numbers of the constituents of the decompositions of the components $f_k^{(1)}(\mathbf{y})$.

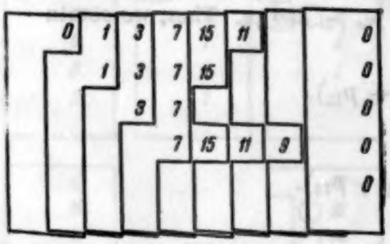

Fig. 4

We thus arrive at the tremendously simple recipe for determining the components $f_k^{(r)}(\mathbf{y})$ of the function $\mathbf{f}^{(r)}(\mathbf{y})$ describing the autonomous system, by the individual possible processes in which the states can vary only in a given sequence, $\mathbf{y}_{\beta_0^{(r)}}, \mathbf{y}_{\beta_1^{(r)}}, \ldots,$

$$\ldots, \mathbf{y}_{\beta_{j_*-1}^{(r)}} \left(\mathbf{y}_{\beta_{j_*}^{(r)}}, \ldots, \mathbf{y}_{\beta_{j_*+m-1}^{(r)}}\right).$$ It suffices for this to choose the cards with the same numbers as the vectors in the given sequence, and to place them one atop the other in accordance with the sequence $\beta_0^{(r)}, \beta_1^{(r)}, \ldots, \beta_{j_*-1}^{(r)}, \beta_{j_*}^{(r)}, \ldots, \beta_{j_*+m-1}^{(r)}$ so that the right side (containing notches or perforations) of each successive card $\beta_{j+1}^{(r)}$ overlaps the left side [containing the numbers $\beta_j^{(r)}$ of the constituent $p_{\beta_j^{(r)}}(\mathbf{y})$] of the previous card $\beta_j^{(r)}$. Then the numbers which will be visible through the notches or perforations of the k-th row of all these cards will be the numbers of the constituents in the decomposition of the k-th component of the function $\mathbf{f}^{(r)}(\mathbf{y})$.

By taking the Boolean sum of the constituents with the numbers read off the k-th row of the cards, we obtain the desired component $f_k^{(r)}(\mathbf{y})$ of the function $\mathbf{f}^{(r)}(\mathbf{y})$. By carrying out this operation for all values of $\underline{k}$ ($k = 1, 2, \ldots, n$), we obtain all the components of the function $\mathbf{f}^{(r)}(\mathbf{y})$ and, by its definition, we obtain thus the function itself.

In order to obtain the function $\mathbf{f}(\mathbf{y})$ describing the autonomous system in which occur not only the process $P(\mathbf{y}_0^{(r)})$ but the complete set of processes given above, it is necessary to carry out a Boolean summation of all N functions found in this way.

In case the given complete set of sequences of state variations in the autonomous system contains a sequence leading to some general sub-sequence, the work necessary for synthesizing this system is shortened,

since the decompositions into constituents found for the components of the functions corresponding to this general sub-sequence may be utilized for determining the constituents into which the functions corresponding to the sequences converging to this sub-sequence are decomposed.

The sequences $P(y_2)$, $P(y_4)$, $P(y_6)$, $P(y_{12})$, and $P(y_{14})$ of our example all lead to one and the same periodic sub-sequence $P(y_0)$ and, consequently, among the constituents of the components $f_k^{(2)}(y)$, $f_k^{(3)}(y)$, $f_k^{(4)}(y)$, $f_k^{(5)}(y)$, and $f_k^{(6)}(y)$ are contained all the constituents, already found by us, in the decomposition of the components $f_k^{(1)}(y)$. Therefore, for the determination of the components $f_k^{(r)}(y)$, where $r = 2, 3, 4, 5$, and 6, it is obviously not necessary to carry out the operations, described above, with the cards corresponding to all the terms of the sequences $P\left(y_{\beta(r)}\right)$. It suffices to carry out these operations only with those cards corresponding to the initial terms of these sequences, including also the first term of their periodic sub-sequence. In particular, for the determination of the components $f_k^{(2)}(y)$ of the function $f^{(2)}(y)$ it is only necessary to use cards 2, 5, and 11. By placing these cards one atop the other by the method described earlier, i.e., as shown in Fig. 5, we get from a consideration of the first row of these cards that, besides the constituents already contained in $f_1^{(1)}(y)$, this component contains the constituents p_2 and p_5, i.e.,

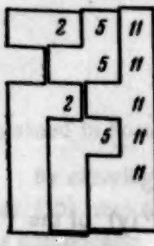

Fig. 5

$$f_1^{(2)}(y) = p_2 + p_5 + f_1^{(1)}(y) = p_2 + p_5 + (p_0 + p_1 + p_3 + p_7 + p_{11} + p_{15}).$$

From a consideration of the remaining three rows of cards 2, 5, and 11, placed as shown in Fig. 5, we obtain, in analogous fashion, that in addition to the constituents contained in $f_2^{(1)}(y)$, $f_3^{(1)}(y)$, and $f_4^{(1)}(y)$, the components $f_2^{(2)}(y)$, $f_3^{(2)}(y)$, and $f_4^{(2)}(y)$ contain just one additional constituent, namely, they contain, respectively, p_5, p_2, and p_5. Thus, we obtain

$$f_2^{(2)}(y) = p_5 + f_2^{(1)}(y) = p_5 + (p_1 + p_3 + p_7 + p_{15}),$$
$$f_3^{(2)}(y) = p_2 + f_3^{(1)}(y) = p_2 + (p_3 + p_7),$$
$$f_4^{(2)}(y) = p_5 + f_4^{(1)}(y) = p_5 + (p_7 + p_9 + p_{11} + p_{15}).$$

By a similar use of cards 4 and 9, we find the components $f_k^{(3)}(y)$:

$$f_1^{(3)}(y) = p_4 + f_1^{(1)}(y), \qquad f_2^{(3)}(y) = f_2^{(1)}(y),$$
$$f_3^{(3)}(y) = f_3^{(1)}(y), \qquad f_4^{(3)}(y) = p_4 + f_4^{(1)}(y).$$

By an analogous use of cards 6, 13, 10, and 1, we obtain the components $f_k^{(4)}(y)$:

$$f_1^{(4)}(y) = p_6 + p_{10} + f_1^{(1)}(y), \qquad f_2^{(4)}(y) = p_{13} + f_2^{(1)}(y),$$
$$f_3^{(4)}(y) = p_6 + f_3^{(6)}(y), \qquad f_4^{(4)}(y) = p_6 + p_{13} + f_4^{(1)}(y).$$

Using cards 12 and 8, we find the components $f_k^{(5)}(y)$:

$$f_1^{(5)}(y) = f_1^{(1)}(y), \qquad f_2^{(5)}(y) = f_2^{(1)}(y),$$
$$f_3^{(5)}(y) = f_3^{(1)}(y), \qquad f_4^{(5)}(y) = p_{12} + f_4^{(1)}(y).$$

Finally, using cards 14 and 9, we find the components $f_k^{(6)}(y)$:

$$f_1^{(6)}(y) = p_{14} + f_1^{(1)}(y), \qquad f_2^{(6)}(y) = f_2^{(1)}(y),$$
$$f_3^{(6)}(y) = f_3^{(1)}(y), \qquad f_4^{(6)}(y) = p_{14} + f_4^{(1)}(y).$$

Taking the union of the decomposition into constituents of the components $f_k^{(r)}(y)$ for all values of $\underline{r}$, we obtain the decomposition into constituents of the components $f_k(y)$ of the desired function $f(y)$:

$$f_1(y) = p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_{10} + p_{11} + p_{14} + p_{15}$$

$$f_2(y) = p_1 + p_3 + p_5 + p_7 + p_{13} + p_{15}, \quad f_3(y) = p_2 + p_3 + p_6 + p_7,$$

$$f_4(y) = p_4 + p_5 + p_6 + p_7 + p_9 + p_{11} + p_{12} + p_{14} + p_{15}.$$

Comparing the values obtained for the components of the function f (y) with the corresponding components of the function f (y) obtained in Example 3 of paper [8], we note that they coincide completely. Consequently, the desired relay system may be implemented by the same relay contact circuit as in the example cited, i.e., by the circuit

$$(y_4' + y_2)\,Y_1 + y_1(y_3 + y_4')\,Y_2 + y_2 y_4' Y_3 + (y_3 + y_1 y_4)\,Y_4.$$

4. Carrying Out the Synthesis of Nonautonomous Relay Systems By Means of Special Cards

Comparing Formulas (6) and (4), we note that the only difference between the first and the second formula is this, that in the (j + 1)-th term $y^{(r)}(j+1)$ of each of the N given sequences, the values of the vector y of the states of the nonautonomous relay system to be synthesized are multiplied, not only by the constituent $p_{\beta_j^{(r)}}(y)$, constructed from the components of the vector y, but also by the constituent $p_{\alpha_j^{(r)}}(x)$, constructed from the components of the vector x of the external stimuli.

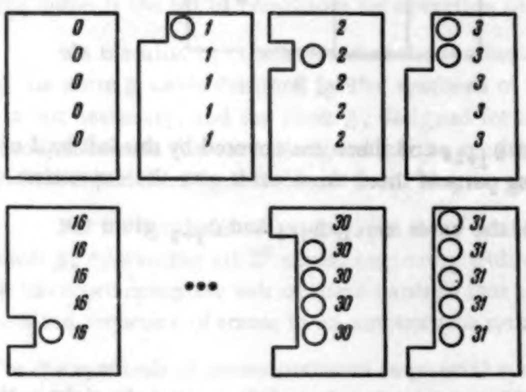

Fig. 6. Cards 0, 1, 2, 3, . . . , 16, . . . , 30, 31, for the synthesis of nonautonomous relay systems.

It follows from this that, for the synthesis of nonautonomous relay systems, it is necessary to have, besides cards β, simulating the values y_β of the vector y and the corresponding constituents, $p_\beta(y)$, cards α simulating the constituents $p_\alpha(x)$. The cards α must contain, at each position, the number α appropriate to the given card. The cards α in no way differ from the right sides of the corresponding cards β, designed for the synthesis of autonomous systems. Briefly put, if the left side were clipped from a card β designed for the synthesis of autonomous systems, we would obtain the card α having the same number, i.e., the card for which $\alpha = \beta$.

The cards β designed for the synthesis of nonautonomous relay systems may have the same construction as the corresponding cards for the synthesis of autonomous systems, but in all cases they must be twice as wide as the latter, though with the same dimensions for the numbers β placed on them. The left sides of the cards β must be twice as wide as the left sides of the same cards designed for the synthesis of autonomous systems. The reason for this is that in the case of nonautonomous system synthesis there must be visible through the notches or perforations of card β_{j+1} two numbers: the number α_j in the corresponding position of card α_j and the number β_j in the card β_j. The right side of the same card, β_{j+1}, must be twice as wide because, in addition to the column containing the number β_{j+1} in all positions, it must contain (for the synthesis of nonautonomous systems) a blank field on which the card α_{j+1} must be placed.

In order that the numbers α_j and β_j which are seen through the notches on card β_{j+1} be easily distinguished from each other, it is preferable to print the numbers on cards α_j and β_j either in a different font or in different colors. If this is too difficult to perform, and the digits of the numbers α_j and β_j are executed in the same font and the same color, then different forms should be used for the apertures on card β for the numbers α and the numbers β; for example, retaining rectangular notches for the numbers β, use circular apertures for the numbers α. Cards β, prepared in this way and designed for the synthesis of nonautonomous relay systems, are shown in Fig. 6.

Figure 7 shows one such card β, namely, card 3, on which has been placed one of the cards α, namely, the one for $\alpha = 5$.

We recall that the numbers β on card β designate the constituent $p_\beta(y)$, and the number placed on card α designates the constituent $p_\alpha(x)$. Therefore, the numbers which are visible on each position of card β, together with those on the card α lying on it, properly designate the product $p_\alpha(x)p_\beta(y)$ of the constituents $p_\alpha(x)$ and $p_\beta(y)$. In particular, the pair of numbers 3 and 5 which is seen at any of the positions of cards $\beta = 3$ and $\alpha = 5$, shown in Fig. 7, designates the product $p_5(x)p_3(y)$ of the constituents $p_5(x)$ and $p_3(y)$.

If the right side of card β_j, with card α_j lying on it, is covered by the left (perforated) side of card β_{j+1} then the total (overlapped) portion of cards α_j, β_j and β_{j+1} gives the expression $p_{\alpha_j}(x)p_{\beta_j}(y)y_{\beta_{j+1}}$. In particular, if cards $\beta_j = 3$ and $\alpha_j = 5$, placed as shown in Fig. 7, are covered by the method just described by card $\beta_{j+1} = 7$, then the total portion of all these cards gives, as shown in Fig. 8, the expression $p_5(x)p_3(y)y_7 = p_5(x)p_3(y)[1, 1, 1, 0, 0] = [p_5(x)p_3(y), p_5(x)p_3(y), p_5(x)p_3(y), 0, 0]$.

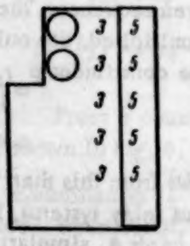

Fig. 7

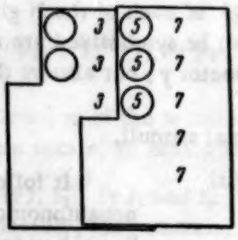

Fig. 8

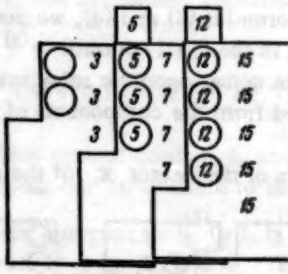

Fig. 9

If some card α_{j+1} is now laid on the right half of card β_{j+1}, and these are covered by the left half of card β_{j+2} in the manner described above, then the overlapping parts of these three cards give the expression $p_{\alpha_{j+1}}(x)\,p_{\beta_{j+1}}(y)\,y_{\beta_{j+2}}$, and the union of the intersection of the cards α_{j+1}, β_{j+1} and β_{j+2} gives the

Boolean sum $p_{\alpha_j}(x)\,p_{\beta_j}(y)\,y_{\beta_{j+1}} + p_{\alpha_{j+1}}(x)\,p_{\beta_{j+1}}(y)\,y_{\beta_{j+2}}$.

In particular, if we lay card $\alpha_{j+1} = 12$ on the right half of card 7 in Fig. 8, and then cover the right half of card 7 plus card $\alpha_{j+1} = 12$ by the left half of card $\beta_{j+2} = 15$, then the overlapping portions of these three cards give the expression $p_{12}(x)\,p_7(y)\,y_{15}$, and the union of the intersection of cards 15, $\alpha_{j+1} = 12$, and 7, and the intersection of cards 7, $\alpha_j = 5$, and 3 gives, as is obvious from Fig. 9, the Boolean sum

$$p_5(x)\,p_3(y)\,y_7 + p_{12}(x)\,p_7(y)\,x_{15}.$$

The first three components of this vector sum, as is obvious from Fig. 9, are identical, and equal the sum $p_5(x)p_3(y) + p_{12}(x)p_7(y)$, and the fourth component equals the expression $p_{12}(x)p_7(y)$.

In order to obtain, by means of cards α_j and β_j, the values of the Boolean sums

$$\sum_{j=0}^{N_r+1} p_{\alpha_j^{(r)}}(x)\,p_{\beta_j^{(r)}}(y)\,y^{(r)}(j+1), \tag{11}$$

which are parts of Formula (6), it is necessary to proceed analogously. It is necessary to take all the cards α_j and β_j corresponding to the terms $x_{\alpha_j^{(r)}}$ and $y_{\beta_j^{(r)}}$ of the given r-th pair of sequences $x^{(r)}(j)$ and $y^{(r)}(j)$, and to place them, one atop the other, in the sequence

$$\beta_0^{(r)},\ \alpha_0^{(r)},\ \beta_1^{(r)},\ \alpha_1^{(r)},\ \ldots,\ \beta_j^{(r)},\ \alpha_j^{(r)},\ \ldots,\ \beta_{N_r-1}^{(r)},\ \alpha_{N_r-1}^{(r)},\ \beta_{N_r}^{(r)} \tag{12}$$

so that the left (perforated) part of each successive card $\beta^{(r)}_{j+1}$ covers the right part of the previous card $\beta^{(r)}_{j}$, together with the card $\alpha^{(r)}_{j}$, lying on the blank field of the latter, i.e., analogously to that shown in Fig. 9, where a fragment of such a disposition of cards α_j and β_j is shown. Then, the numbers which are visible through the the rectangular notches and circular apertures in cards $\beta^{(r)}_{j+1}$ will model the vectors $p_{\alpha^{(r)}_j}(x)\, p_{\beta^{(r)}_j}(y)\, y^{(r)}(j+1)$, and the entire set of pairs of numbers $\beta^{(r)}_{j}$ and $\alpha^{(r)}_{j}$ visible through the rectangular notches and circular apertures of all $N_r + 1$ cards, positioned in the way described, models the Boolean sum of all these vectors, i.e., Sum (11).

By taking the Boolean sum of the paired products $p_{\alpha^{(r)}_j}(x)\, p_{\beta^{(r)}_j}(y)$ of constituents, where the numbers $\beta^{(r)}_{j}$ and $\alpha^{(r)}_{j}$ are visible through the rectangular notches and circular apertures of the k-th row of cards $\beta^{(r)}_{j+1}$ placed according to the method described above and interwoven with cards α_j in Sequence (12), we obtain the k-th components of the vector sums in (11).

If we carry out the operations described above on all N given pairs of sequences $x^{(r)}(j)$ and $y^{(r)}(j)$, and find the union of the results obtained, we get the right side of Formula (6), i.e., the values of the desired function $f(x, y)$. Each k-th component of this function is the Boolean sum of the k-th components of all N Boolean vector sums (11), i.e., the union of all paired products of the constituents $p_{\alpha^{(r)}_j}(x)\, p_{\beta^{(r)}_j}(y)$ the numbers of which may be read through the apertures of the k-th rows of cards $\beta^{(r)}_{j+1}$ in all the sequences of (12). By definition, this union is the set of conditions for operation of relay Y_k in the nonautonomous system to be synthesized.

We mention in conclusion that the synthesis of autonomous relay systems can be carried out with the aid of the same β cards designed for the synthesis of nonautonomous sequential relay systems. For this, the cards α are not necessary, and the cards β, designed for the synthesis of nonautonomous systems, are used in exactly the same manner as the cards designed for the synthesis of only autonomous systems. Here the circular apertures in the cards and the blank fields on them play no role, and no attention should be paid to them.

For the synthesis of autonomous relay systems, it is necessary to have no less than one complete set of cards β, containing all 2^n cards, beginning with card 0 and ending with card $2^n - 1$. However, it is convenient to have two complete sets of these cards, so that some periodic sub-sequence, one of which terminates every possible sequence of states in an autonomous system, be terminated by the same card $\beta^{(r)}_{j_0}$ with which it began. For the synthesis of nonautonomous sequential systems, in addition to two complete sets of cards β, designed for the synthesis of these systems, it is necessary to have several complete sets of the cards α. For convenience in changing cards α upon going from one sequence of type (12) and its corresponding disposition of cards β_j and α, to another, it is better to make the cards α somewhat longer than the cards β so that they will project beyond the edge of cards β, as shown in Fig. 9.

Sets of cards designed for the synthesis of relay systems constructed of $\underline{n}$ relays and depending on $\underline{m}$ independent variables may also be used for the synthesis of systems constructed of fewer than $\underline{n}$ relays and dependent on less than $\underline{m}$ variables. In the examples considered above, the cards that were designed for five relays and five variables were used for the synthesis of autonomous systems constructed with four relays, and for the synthesis of nonautonomous systems containing four relays and depending on four independent variables.

The punched card method of synthesizing sequential relay systems may be used with manual placement of the cards and visual reading of the numbers through the apertures in the cards for the synthesis of relay systems where the total number of variables (dependent and independent) does not exceed 8 or 9.

If mechanized, the punched card method could apparently accommodate twice the total number of relays and independent variables in the systems to be synthesized.

To use the punched card method in practice, it is necessary to provide for the most efficient construction of the cases in which the cards α and β are to be stored, the construction of the framework on which the assembly of sequences of type (12) is to be carried out, the construction of the cards themselves to guarantee rapid assembly and storage of the assembled sequences of cards, etc.

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ON THE STABILITY OF PERIODIC REGIMENS IN NONLINEAR SYSTEMS WITH PIECEWISE LINEAR CHARACTERISTICS

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A way is described which allows one to find the equations of a linear approximation solving the problem of stability of a periodic solution in a system with a piecewise linear characteristic.

1. As in works [2, 3], we consider an automatic control system described by the equations

$$\dot{z}_i = \sum_{j=1}^{n} a_{ij} z_j + \lambda_i f(z_1) + F_i(t) \quad (i = 1, \ldots, n), \tag{1}$$

where a_{ij} and λ_i are given numbers (some of which are zero), $F_i(t)$ are periodic functions with common period τ (in particular, constants, including zero) and $f(z_1)$ is the given piecewise linear function.

For concreteness, let the transitions from one line in the characteristic $f(z_1)$ to another occur at the moments when z_1 first attains the values σ_α ($\alpha = 1, \ldots, r$). We assume that there has been found a periodic solution of Equations (1) with period τ, for example, as cited in [2, 3], and that it is required to investigate its stability (in the sense of Liapunov). For this purpose, the theorem proved in [4] may be used. This theorem generalizes to systems of differential equations with discontinuous right members, the classical theory of stability by linear approximations, developed by Liapunov, applicable to systems of differential equations with analytic right members.

2. Before getting back to System (1), we first consider the more general equations

$$\dot{z}_i = f_i(z_1, \ldots, z_n, t) \quad (i = 1, \ldots, n), \tag{2}$$

where all the f_i are functions periodic in t with common period τ, or functions not depending upon t.

Let $(z_1, \ldots, z_n, t)$ space be sectioned by the surfaces $F_\alpha(z_1, \ldots, z_n, t) = 0$ into regions H_α ($\alpha = 1, 2, \ldots,$). It is assumed that in each region H_α, the given curves are sufficiently smooth; upon a transition through a surface F_α, there may occur discontinuities both in the functions f_i, themselves and in their partial derivatives. The integral curves of Equations (2) are continuous, however, but are subject to breaks (discontinuities in the derivatives) at the surfaces F_α.

Further, let $z_i = \tilde{z}_i(t)$ be the periodic solutions of System (2) intersecting the surfaces $F_\alpha = 0$ at the moments $t = t_\alpha$.

Together with System (2), we consider the linear equations, the right members of which are given for all regions H_α:

$$\dot{x}_i = \sum_{j=1}^{n} b_{ij} x_j \quad (i = 1, \ldots, n), \tag{3}$$

where $b_{ij} = \left[\frac{\partial f_i}{\partial z_j}\right]_{z=\tilde{z}(t)}$ are periodic coefficients with common period τ, defined for arbitrary $\underline{t}$ except $t = t_\alpha$.

We add to Equations (3) linear relationships defining the discontinuities in x_i for each moment $t = t_\alpha$:

$$x_i(t_\alpha + 0) - x_i(t_\alpha - 0) = \xi_i \sum_{k=1}^{n} h_k^- x_k(t_\alpha - 0) = \xi_i \sum_{k=1}^{n} h_k^+ x_k(t_\alpha + 0), \tag{4}$$

where

$$h_k^{\pm} = \left[\frac{\frac{\partial F_\alpha}{\partial z_k}}{\left(\frac{dF_\alpha}{dt}\right)^{\pm}}\right]_{z=\tilde{z}(t_\alpha)}$$

Equations (3) and Relationships (4) jointly determine the discontinuities (for $t = t_\alpha$) in the integral curves. We call the set of Equations (3) and Relationships (4) the linear approximation of Equations (2) for their periodic solutions $z_i = \tilde{z}_i(t)$. Then, with certain additional limitations* imposed on the functions f_i and F_α, we have the following theorem** [1]:

If the null solution $x_i = 0$ of the linear approximation system(3) and (4), is asymptotically stable, then the periodic solutions $z_i = \tilde{z}_i(t)$ of Equations (2) are also asymptotically stable.

3. Returning to System (1), we note that if, in the interval $t_{\alpha-1} < t < t_\alpha$, the section of the characteristic has the equation

$$f(z_1) = k_\alpha z_1 + s_\alpha,$$

then Equations (3) for System (1) may be written in the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \lambda_i\, l(t)\, x_1, \tag{3'}$$

where $l(t)$ is the periodic piecewise constant function

$$l(t) = k_\alpha \quad (t_{\alpha-1} < t < t_\alpha,\ \alpha = 1, 2, \ldots).$$

The equations for the discontinuity surfaces for Equations (1) we write in the form $z_1 - \sigma_\alpha = 0$.

Therefore, Conditions (4) lead to the conditions

$$x_i(t_\alpha + 0) - x_i(t_\alpha - 0) = R_i^\alpha x_1(t_\alpha - 0), \tag{4'}$$

where

$$R_i^\alpha = \epsilon \lambda_i \xi \frac{1}{\dot{z}_1(t_\alpha - 0)}.$$

Here ξ is the magnitude of the discontinuity of the function $f(z_1)$ for $z_1 = \tilde{z}_1(t_\alpha)$, and $\epsilon = 1$ or $\epsilon = -1$ depending upon the direction of intersection of the surface by the trajectory.

Integrating Equations (3') for $t_{\alpha-1} < t < t_\alpha$, and adjusting them at the limits of the period by taking (4') into account, we find the linear relationships expressing the coordinate values at the end of the period, $x_i(t_0 + \tau)$, in terms of their values at the beginning of the period, $x_i(t_0)$.

We write these relationships in matrix form thusly:

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* These limitations are not discussed here since they are always satisfied for System (1) which is of interest us to.

** The analogous theorem on instability is also proven (see [1]).

$$x(t_0+\tau)=Ux(t_0),$$

where U is the constant transformation matrix.

Thus, for the stability of the periodic solution being considered, it suffices to require that the characteristic numbers of matrix U, i.e., the roots of the characteristic equation $\det(U-\lambda E)=0$ be distributed in the unit circle.

4. Certainly, the question of stability of the periodic solution of System (1) could be solved by direct adjustment. For this, Equations (1) would have to be integrated over a period, with individual integrations necessary for each step $t_{\alpha-1}<t<t_\alpha$, to determine in this way the transformation of the quantities $z_i(t_0)$ to $z_i(t_0+\tau)$ ($i=1,\ldots,n$), and to investigate the stability of the invariant points of the transformation corresponding to the periodic solutions. Such an investigation, in its turn, requires the expansion of the solution obtained in a series of powers of the variation of the initial deviation and of time t_α, a discarding of the nonlinear terms, the elimination of the variation in time t_α, and, finally, the determination of the transformation matrix of the coordinate variation during the period.

The method described above, using the theorem on the linear approximation for piecewise-smooth systems, allows this computational work to be shortened, since it allows the introduction of the variation of time t_α to be avoided, and allows the immediate writing of the necessary linear relationships

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* See English translation.

THE TRANSFER FUNCTION OF A DC MOTOR CONTROLLED BY VARYING THE EXCITATION VOLTAGE

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The transfer function is derived for a dc motor with a supplementary series excitation winding.

In many systems of automatic regulation and control, dc motors controlled by varying the excitation voltage are employed. For computing and investigating the stability of such systems, an expression for the transfer function of the motor, well known in the literature (for example, [1]) is generally used. This expression, and the corresponding schematic design in which the motor takes the form of a network with standard links and rigid connections between them, are correct in the cases when the motor has one excitation winding. In many cases though, an additional series excitation winding is employed in order to guarantee the required starting characteristics. The transfer function for such cases cannot be assumed to be the same as for motors without the series winding.

Below, an expression is derived for the transfer function of a motor which, in addition to the electromechanical time constant and the time constants of the excitation winding circuit and the armature circuit, takes into account the action of the series excitation winding, and the corresponding schematic circuit is given.

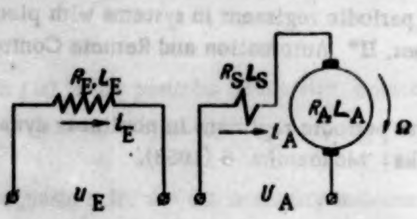

Fig. 1

The circuit for switching on the motor is given in Fig. 1. The motor armature is connected to a dc voltage source of magnitude U_A.

For the circuits of the basic excitation winding and the motor armature, we may write the following equations:

$$u_E = i_E R_E + w_E \frac{d\Phi'}{dt} \tag{1}$$

$$U_A = e + i_A(R_A + R_S) + L_A \frac{di}{dt} + w_S \frac{d\Phi'}{dt} \tag{2}$$

The equation for the motor shaft moment has the form:

$$J\frac{d\Omega}{dt} = M_M - M_S \tag{3}$$

In these equations the following nomenclature was used: u_E is the voltage on the terminals of the basic excitation winding, i_A and i_E are the currents in the armature and basic excitation winding, $\underline{e}$ is the counter-emf, Ω is the nominal speed of armature rotation, M_M and M_S are the turning moment of the motor and the impedance moment, w_E and w_S are the numbers of turns in the basic and series excitation windings, R_A, R_E, and R_S are the winding resistances of the armature, basic excitation winding and series winding, L_A is the inductance of the armature winding, J is the armature's moment of inertia and Φ is the total flux linked with the basic and series excitation windings.

For the turning moment and counter-emf we may write

$$M_M = k_1 \Phi i_A \tag{4}$$

$$e = k_2 \Phi \Omega. \tag{5}$$

In Equations (4) and (5), Φ is the effective resulting flux threading the armature winding. The total and the effective flux are related by the expression [2]

$$\Phi' = \Phi [1 + \gamma_s (\sigma - 1)], \tag{6}$$

where σ is the leakage coefficient of the poles and γ_s is the coefficient of leakage flux linkage with the excitation winding. Taking $\alpha = \gamma_s (\sigma - 1)$, we may write

$$\Phi' = \Phi (1 + \alpha). \tag{6'}$$

Replacing the magnetization curve of the motor by a rectilinear characteristic, and ignoring the reaction of the armature and the eddy currents in the massive parts of the system, we write the expression for the total flux in the form

$$\Phi' = \frac{L_E}{w_E} i_E + \frac{L_S}{w_S} i_A, \tag{7}$$

where L_E and L_S are, respectively, the inductances of the basic and the series excitation windings.

Since it is assumed that there is no leakage between the basic and the series excitation windings, we have that $L_S = L_E \left(\frac{W_S}{W_E}\right)^2$.

We write the differential equations for the motor in incremental form, expressing the variables u_E, i_E, i_A, Ω, and Φ in terms of deviations from the steady-state values:

$$u_E = u_{E_0} + \Delta u_E, \; i_E = i_{E_0} + \Delta i_E, \; i_A = i_{A_0} + \Delta i_A,$$
$$\Omega = \Omega_0 + \Delta\Omega, \; \Phi = \Phi_0 + \Delta\Phi.$$

The incremental equations, in operator form, will be:

$$\Delta u_E = \Delta i_E R_E + w_E (1 + \alpha) p \Delta\Phi, \tag{8}$$

$$0 = k_2 (\Phi_0 \Delta\Omega + \Omega_0 \Delta\Phi) + \Delta i_A (R_A + R_S) + L_A p \Delta i_A + w_S (1 + \alpha) p \Delta\Phi, \tag{9}$$

$$J p \Delta\Omega = k_1 (\Phi_0 \Delta i_A + \Delta\Phi i_{A_0}). \tag{10}$$

$$(1 + \alpha) \Delta\Phi = \frac{L_E}{w_E} \Delta i_E + \frac{L_S}{w_S} \Delta i_A. \tag{11}$$

From Equations (9) and (10) we obtain the differential equation relating the incremental armature current with the incremental speed

$$\left[k_2 p \Omega_0 + p^2 (1 + \alpha) w_S + \frac{k_1 k_2 \Phi_0}{J} i_{A_0}\right] \Delta\Omega =$$
$$= \frac{\Phi_0 k_1}{J}\left[k_2 \Omega_0 + p (1 + \alpha) w_S - \frac{i_{A_0} (R_A + R_S + p L_A)}{\Phi_0}\right] \Delta i_A. \tag{12}$$

We introduce the notation: $T_M = \frac{J (R_A + R_S)}{k_1 k_2 \Phi_0^2}$ is the electromechanical time constant of the motor, $T_A = \frac{L_A}{R_A + R_S}$ is the time constant of the armature winding, $T_S = \frac{L_S}{R_A + R_S}$ is the time constant of the

series winding, $T_{\text{н}} = \frac{w_S}{k_2 \Omega_0}$, $\varkappa = \frac{T_S}{T_{\text{н}}}$, $\varepsilon = \frac{i_{A_0}(R_A + R_S)}{U_A}$ is the relative voltage drop in the armature circuit.

Using this notation we may write, instead of (12), the following equation:

$$\left\{\frac{\varepsilon}{1-\varepsilon} + pT_{\text{м}}[1 + (1+\alpha)\,pT_{\text{н}}]\right\}\frac{\Delta\Omega}{\Omega_0} = \frac{\varepsilon}{1-\varepsilon}\left\{\frac{1-(pT_A+2)\,\varepsilon}{1-\varepsilon} + (1+\alpha)\,pT_{\text{н}}\right\}\frac{\Delta i_A}{i_A}. \tag{13}$$

We also obtain from Equations (9) and (10) the equation relating the incremental effective flux with the incremental armature current:

$$[1 + pT_{\text{м}}(1 + pT_A)]\frac{\Delta i_A}{i_{A_0}} = -\left\{1 + \frac{1-\varepsilon}{\varepsilon}\,pT_{\text{м}}[1 + (1+\alpha)\,pT_{\text{н}}]\right\}\frac{\Delta\Phi}{\Phi_0}. \tag{14}$$

Finally, from Equations (8), (11), and (14) we obtain the relationship between the incremental excitation voltage and excitation current:

$$\left\{(1 + pT_E)[1 + pT_{\text{м}}(1 + pT_A)] + \varkappa pT_{\text{м}}[1 + (1+\alpha)\,pT_{\text{н}}] + \varkappa\frac{\varepsilon}{1-\varepsilon}\right\}\Delta i_E =$$
$$= \frac{1}{R_E}\left\{1 + pT_{\text{м}}(1 + pT_A) + \varkappa pT_{\text{м}}[1 + (1+\alpha)\,pT_{\text{н}}] + \varkappa\frac{\varepsilon}{1-\varepsilon}\right\}\Delta u_E \tag{15}$$

On the basis of Equations (11), (13), (14), and (15), it is possible to set up the structural schematic for the motor (Fig. 2).

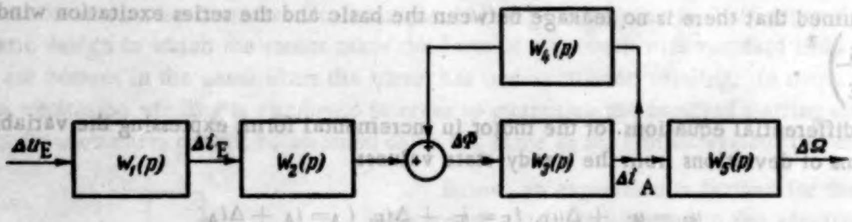

Fig. 2

Since the action of the series winding was taken into account, the structural schematic for the motor is more complicated than the one obtained when only one excitation winding is present. The structural schematic expresses the fact that when the additional series winding is present in the motor, it is impossible to present the circuit as one consisting of standard links connected in series. The transfer functions of the links in the structural schematic shown in Fig. 2 have the following expressions:

$$W_1(p) = \frac{\Delta i_E}{\Delta u_E} = \frac{1}{R_E}\,\frac{1 + pT_{\text{м}}(1 + pT_A) + \varkappa pT_{\text{м}}[1 + (1+\alpha)\,pT_{\text{н}}] + \varkappa\frac{\varepsilon}{1+\varepsilon}}{(1 + pT_E)[1 + pT_{\text{м}}(1 + pT_A)] + \varkappa pT_{\text{м}}[1 + (1+\alpha)\,pT_{\text{н}}] + \varkappa\frac{\varepsilon}{1-\varepsilon}}, \tag{16}$$

$$W_2(p) = \frac{L_E}{w_E}\,\frac{1}{1+\alpha}, \tag{17}$$

$$W_3(p) = \frac{\Delta i_A}{\Delta\Phi} = -\frac{i_{A_0}}{\Phi_0}\,\frac{1 + \frac{1-\varepsilon}{\varepsilon}\,pT_{\text{м}}[1 + (1+\alpha)\,pT_{\text{н}}]}{1 + pT_{\text{м}}(1 + pT_A)}, \tag{18}$$

$$W_4(p) = \frac{L_S}{w_S}\,\frac{1}{1+\alpha}, \tag{19}$$

$$W_5(p) = \frac{\Delta\Omega}{\Delta i_A} = \frac{\Omega_0}{i_{A_0}}\,\frac{\varepsilon}{1-\varepsilon}\,\frac{\frac{1+(pT_A+2)\,\varepsilon}{1-\varepsilon} + (1+\alpha)\,pT_{\text{н}}}{\frac{\varepsilon}{1-\varepsilon} + pT_{\text{м}}[1 + (1+\alpha)\,pT_{\text{н}}]}. \tag{20}$$

The over-all transfer function of the motor, if we consider the input coordinate to be the voltage in the excitation circuit and the output coordinate to be the nominal speed, has the expression

$$W_{\mathrm{M}}(p) = \frac{\Delta\Omega}{\Delta u_{\mathrm{E}}} =$$

$$= -\frac{T_{\mathrm{E}}\Omega_0}{W_{\mathrm{E}}\Phi_0} \frac{\frac{1-(pT_{\mathrm{A}}+2)\varepsilon}{1-\varepsilon} + (1+\alpha)\,pT_{\text{н}}}{(1+pT_{\mathrm{E}})\,[1+pT_{\text{м}}(1+pT_{\mathrm{A}})] + pT_{\text{м}}[1+(1+\alpha)\,pT_{\text{н}}]\,\varkappa + \varkappa\frac{\varepsilon}{1-\varepsilon}}. \tag{21}$$

If the regimen of operation of the motor is varied, a host of quantities entering into (21) have their values changed. The quantities dependent on the mode of operation are the nominal speed Ω_0, the effective flux Φ_0, the electromechanical time constant T_{M}, and the relative voltage drop across the armature ε.

It is possible to detect the influence of a series excitation motor winding on system stability in the example of a system for stabilizing the frequency ν of the output voltage of an electromagnetic transformer, consisting of a dc motor and an ac generator. Figure 3 gives a block schematic of the electromagnetic transformer with frequency stabilization.

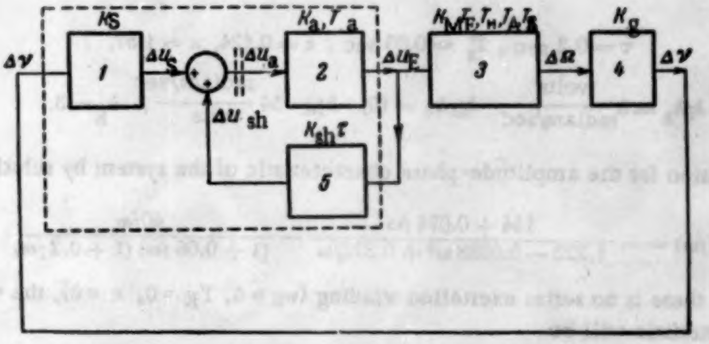

Fig. 3. 1) is the measuring organ; 2) is the amplifier; 3) is the motor; 4) is the generator and 5) is a differentiator.

For setting up the differential equations for the system with frequency stabilization, we consider that the measuring organ is noninertial, but the amplifier is an inertial link with gain of k_a and time constant T_a. In the motor, for simplicity, we ignore the electromagnetic time constant of the armature ($T_A = 0$) and the pole leakage ($\alpha = 1$). In this case, the expressions for the transfer functions of the system links take the following form:

measuring organ

$$W_S(p) = k_S,$$

amplifier

$$W_a(p) = \frac{k_a}{1+pT_a},$$

motor

$$W_M(p) = -k_M \frac{\frac{1-2\varepsilon}{1-\varepsilon} + pT_{\text{н}}}{(1+pT_{\mathrm{E}})(1+pT_{\text{м}}) + pT_{\text{м}}(1+pT_{\text{н}})\,\varkappa + \varkappa\frac{\varepsilon}{1-\varepsilon}},$$

generator

$$W_g(p) = k_g,$$

differentiator

$$W_{sh}(p) = -\frac{k_{sh} p\tau}{1 + p\tau}.$$

The transfer function of the system, opened at the amplifier input, has the form

$$W(p) = W_a(p) W_S(p) W_M(p) W_g(p) + W_a(p) W_{sh}(p) \tag{22}$$

or

$$W(p) = -\frac{k_a k_S k_M k_g}{1 + pT_a} \frac{\frac{1 - 2\varepsilon}{1 - \varepsilon} + pT_K}{(1 + pT_E)(1 + pT_M) + pT_M(1 + pT_K)\varkappa + \varkappa\frac{\varepsilon}{1 - \varepsilon}} - \frac{k_a k_{sh} p\tau}{(1 + pT_a)(1 + p\tau)}. \tag{23}$$

In the example considered, $T_E = 0.0683$ sec, $T_M = 0.118$ sec, $T_K = 0.00044$ sec,

$$\tau = 0.2 \text{ sec}, \; T_a = 0.06 \text{ sec}, \; \varepsilon = 0.124, \; \varkappa = 1.57,$$

$$k_S k_a = 3\,\frac{\text{volts}}{\text{radians/sec}}, \; k_{sh} k_a = 10, \; k_M = 54\,\frac{\text{radians/sec}}{\text{volts}}, \; k_g = 3.$$

We get the expression for the amplitude-phase characteristic of the system by substituting $p = j\omega$ in (23):

$$W(j\omega) = -\frac{144 + 0.074\,j\omega}{1{,}222 - 0.0088\,\omega^2 + 0.372 j\omega} - \frac{40 j\omega}{(1 + 0.06\,j\omega)(1 + 0.2\,j\omega)}. \tag{24}$$

In the case where there is no series excitation winding ($w_S = 0$, $T_K = 0$, $\kappa = 0$), the expression for the amplitude-phase characteristic will be

$$W(j\omega) = -\frac{144}{(1 + 0.0683\,j\omega)(1 + 0.118\,j\omega)} - \frac{40\,j\omega}{(1 + 0.06\,j\omega)(1 + 0.2\,j\omega)}. \tag{25}$$

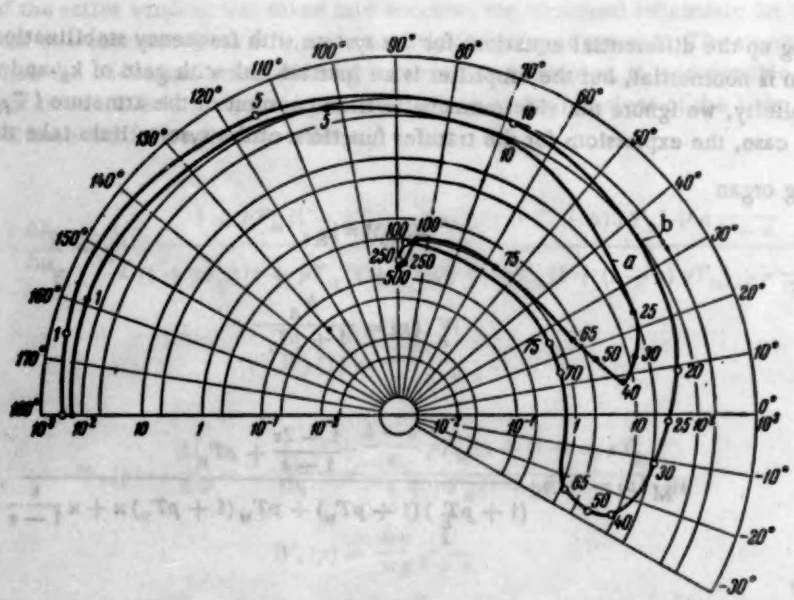

Fig. 4

Figure 4 gives, on a logarithmic scale, the amplitude-phase characteristics computed on the basis of (24) and (25): *a* is with a series excitation winding present, *b* is without the series winding. The curves of Fig. 4 show that the series excitation winding of the motor allows stable system functioning to be achieved. For the given values of the system parameters, the system will be unstable in the absence of the series winding.

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ALL-UNION CONFERENCE ON MAGNETIC ELEMENTS IN AUTOMATION, REMOTE CONTROL AND COMPUTING TECHNOLOGY

During November 25-30, 1957, in Moscow, there was held an All-Union Conference on magnetic elements in automation, remote control and computing technology, organized by the Institute for Automation and Remote Control of the Academy of Sciences, USSR, and by the Board (Komissiia) for magnetic amplifiers and contactless magnetic elements. The aim of the Conference was the discussion of the state of scientific problems in the theory, design and application of magnetic elements in automation, remote control and computing technology.

More than 800 participants took part in the Conference, representing 246 organizations, including the institutes of the Academy of Sciences, USSR, the Academy of Science, Ukrainian SSR, the Azerbaidzhan SSR, the Latvian SSR, the Ural branch of the Academy of Sciences, USSR, research and educational institutes, building and planning organizations of various branches of industry. Representatives of the Czechoslovakian Academy of Science, the Polish Academy of Science, the Bulgarian Academy of Science and the Chinese Academy of Science participated in the Conference.

Seventy papers and communications were presented, drawing lively interest from the Conference participants. More than forty people came forward in the discussions. The work of the Conference was carried out in two sections: magnetic amplifiers and contactless magnetic elements.

B. S. Sotskov, President of the Board for magnetic amplifiers and contactless magnetic elements, mentioned in his opening words that magnetic amplifiers and contactless magnetic elements play a large role in the development of the technological means of automation. However, the scope and tempo of their production and use still do not satisfy the requirements of our rapidly developing national economy.

B. S. Sotskov, in his paper, "Modern state and problems of the development of magnetic elements for automation, remote control, and computing technology," further stated that magnetic amplifiers and semiconductor devices do not compete with one another. The appearance of crystal triodes not only did not limit the domain of application of magnetic elements and, in particular, of magnetic amplifiers but, on the contrary, created favorable conditions for their application. The speaker backed up his opinion with many examples of new uses for magnetic amplifiers and of their use in conjunction with semiconductors. In conclusion, the speaker voiced his surmise that, as a result of the Conference, there would be a significant expansion in the necessary work by the organization of extensive production of various types of magnetic elements.

In his paper, "Dynamic characteristics of electrical circuit elements," K. M. Polivanov presented a new method of representing dynamic characteristics in the form of hypersurfaces. The use of actual dynamic characteristics allows the computation of surface effects to be carried out approximately, taking into account viscosity, the influence of which in materials with approximately rectangular hysteresis loops is dominant in regions of strong fields.

R. V. Telesnin gave a paper on the subject "On the influence of magnetic viscosity on the processes of core switching."

In his paper, "Certain factors influencing the static and dynamic characteristics of toroidal cores," M. A. Rozenblat showed the influence on the static characteristics of magnetic amplifiers of eddy currents and the geometric relationships of the cores.

During November 26-29, the work of the Conference was carried out by sections.

A. Section on Magnetic Amplifiers

The paper by N. P. Vasil'eva and O. A. Sedykh, "Unified method of designing magnetic power amplifiers," was devoted to the design of magnetic amplifiers with output power from several milliwatts to tens of kilowatts. As the basis of their calculations the authors placed the functional dependence of the minimum specific volume of the cores, i.e., the volume per unit power under optimum working conditions, on the strength of the magnetizing field. This dependence is determined solely by the cores' magnetic characteristics and, for various designs of the magnetic amplifiers, can be expressed by equations differing only in their constant coefficients. The optimal geometry of the cores for magnetic amplifiers is determined from conditions for obtaining minimum weight, or dimensions, or cost of the amplifier. The simple engineering method presented allows magnetic amplifiers to be designed according to arbitrary schemes, including ac amplifiers and high-speed amplifiers.

In his paper, "On the design of ac choked magnetic amplifier circuits," N. A. Kaluzhnikov dwelled on the question of designing choked magnetic amplifiers with active inductive and complex ac loading, and also for the various forms of loading connected to the rectifier output. The speaker presented graphic computations for various loads and also for various loading characteristics of the magnetic amplifier.

N. N. Aleksandrov, in the paper, "On the geometric locus of points on the simultaneous magnetization curves determining the functioning of a two-stage magnetic amplifier with constant load," extended the method of computing the characteristic of a single-stage amplifier by means of a loading ellipse to two-stage magnetic amplifiers.

L. V. Safris gave a paper on the subject "Computation of magnetic amplifiers by linearized magnetization characteristics." The magnetization characteristics of saturated choke cores are, by this method, described by linearized equations in which the influence of the constant and the variable components of the magnetic field are taken into account. The equations obtained render possible the establishment of the connection between the controlling current and the current and emf in the ac winding as well as the input inductance of the saturated choke. Conversion schemes were given for various types of magnetic amplifiers. The method of linearization allows uniform computation for the steady-state and the transient regimens in a number of basic circuits for magnetic amplifiers with cores of various properties.

T. Kh. Stefanovich, in his paper, "On the design of self-saturating magnetic amplifiers," presented a theory of functioning of magnetic amplifiers with internal feedback connections (amplistats) without taking into account the width of the hysteresis loop of the core material, and also gave a design method guaranteeing the functioning of the amplifier on the linear portion of its characteristic. The design is carried out using specimen experimental characteristics of an amplifier prepared from the given magnetic material according to the same design as the planned amplifier.

In the paper of V. L. Benin and I. N. Senkevich on the subject "Special features in the design of magnetic amplifiers in reversing circuits for controlling short-circuited, three-phase asynchronous motors," the method, developed by the authors, was presented for computing the parameters of magnetic amplifiers, taking into account all the fundamental peculiarities of their functioning in three-phase systems with the condition of minimum outlay of active materials. Based on the method of symmetric components, quantitative relationships were obtained characterizing the functioning of magnetic amplifiers in various circuits.

The paper of R. Kh. Bal'ian, "Engineering design of chokes for magnetic amplifiers with feedback," was devoted to the design of magnetic amplifiers with feedback for a given load, guaranteeing minimum dimensions, weight or cost. The design method is applicable both to single-stage and to two-stage amplifiers with arbitrary load characteristics. The fundamental regularities of optimal core geometry were given, and the peculiarities of the thermal regimen in choked magnetic amplifiers were considered, as well as the choice of the admissible current density in the windings.

In the paper of M. A. Boiarchenkov and N. P. Vasil'eva "Fast-acting magnetic amplifiers," the functioning of the fundamental designs for fast-acting magnetic amplifiers was analyzed, and the conditions were derived for obtaining fast action in arbitrary amplifier designs; a series of designs of magnetic amplifiers illustrating the application of these conditions was presented. The authors produced experimental data and approximate computations for a fast-acting amplifier with a dc output, and also data for the conversion of ordinary designs with dc output to fast-acting ones.

V. I. Molotkov, in the paper "Investigation of the functioning of a magnetic amplifier with a capacitor filter at the output of a rectifier bridge," using analytic methods for investigating the functioning of a magnetic amplifier in the transient and steady states, derived the equation for determining the boundaries of the possible regimens of functioning of the amplifier.

N. N. Aleksandrov presented a communication entitled "Peculiarities of functioning of a magnetic amplifier with the load connected through a rectifier bridge and inductively connected to short-circuited coils." As such short-circuited coils appear the contours of the eddy currents in the massive magnetic circuits of electrical machines, their commutating sections, or the working contours of the windings in bridge magnetic amplifiers in successive cascades.

The communication of A. S. Bogoslovskii,"Saturable reactors in circuits controlled by high-power three-phase semiconductor rectifiers," gave the analysis of circuits of three-phase semiconductors with saturated chokes, and the results of experimental tests on them.

In the paper of A. L. Pisarev, "Investigation of a magnetizable reactor with a rectifier load circuit, controlled from a dc source," there were investigated the processes leading to the saturation of the choke by purely active loads with rectifiers in the load circuit, and the functioning of the choke was investigated for approximate magnetic characteristics of the material, considering it as being three segments, and as being a parallelogram. The author gave an analysis of the functioning of the saturated choke with complex loading. A method was given for computations, allowing a step-by-step construction of the transients for a falling load current.

In the paper of A. A. Golovan, "Magnetic amplifiers with high-ohmic loads," there were considered designs of magnetic amplifiers working effectively with loads on the order of tens and hundreds of thousands of ohms; with a source supplying voltage pulses alternating in sign, there was investigated a one-half-period nonreversing amplifier design, and results of experimental investigation of this design were given.

V. I. Minkevich, in his communication, "Certain questions in designing summing magnetic amplifiers with low sensitivity thresholds," presented a method for reducing zero drift in a differential magnetic amplifier by introducing an active impedance into one of its arms. The speaker also considered the possibility of employing one-half-period circuits in summing amplifiers. A method was presented for sorting cores according to their magnetic properties by the use of a series oscillograph.

In the paper of A. M. Bamdas, V. A. Somov, and A. O. Shmidt, "Transformers and stabilizers, regulated by magnetizable shunts," there was given a sketch of the principles of designing and building regulatable transformers by means of shunts, considered as elements in the theory of transformer regulation, and data were provided for the carrying out of the construction.

I. G. Gol'dreer and Iu. V. Afanas'ev presented the paper "The theory of even-harmonic (phase-pulse) magnetic amplifiers," in which were considered the specific peculiarities of functioning of even-harmonic magnetic amplifiers from the point of view of the general theory of amplifiers. Presented in the paper were methods of selecting regimens of functioning, and basic parameters of magnetic amplifiers.

V. A. Oleinikov, in the paper "Investigation of the work of magnetic amplifiers in sampled-data control systems," employed a graphico-analytic method of computation for the construction of the transient response, and dealt with the possibility of obtaining forced transient responses. The pulse elements in the magnetic amplifier are here considered as equivalent to inertial links. The author showed that the stability and speed of action of a servosystem, working in a pulse regimen, are increased in comparison with ordinary servosystems.

F. I. Kerbnikov and M. A. Rozenblat, in the paper "Magnetic modulators with lateral excitation," gave a method for designing modulators with mutually perpendicular fields, working either on the principle of frequency doubling or with the output at the fundamental frequency of the supply source. Such modulators have high sensitivity and stability, the transient response in them lasting for one period of the source frequency. The authors made recommendations on the use of modulators in devices for automation.

V. S. Volodin and G. V. Subbotina, in the paper "Modern applications of semiconducting triodes and magnetic amplifiers," gave a survey of the existing devices with the contemporary application of semiconducting triodes (transistors) and magnetic amplifiers, and gave an analysis of a design for phase-sensitive rectifiers for the cases when certain types of magnetic amplifiers are used. The possibility was demonstrated of increasing output power and speed of action by the modern application of transistors and magnetic amplifiers.

R. A. Lipman, in the paper "Magnetic amplifier control by means of transistors," gave computed relationships in fast-acting magnetic amplifiers, controlled by transistors, and provided several circuits for cascade connections of transistors and magnetic amplifiers.

B. A. Mitrofanov and N. I. Chicherin, in the paper "Applications of semiconductor diodes and triodes in amplifiers of servosystems," dealt with the most widely used elements of servosystems, their circuit connections, and also provided data on servosystems with type ADP motors with powers of 4 and of 50 watts.

G. I. Shapkaits, in the paper "Experiments in the use of two-stage magnetic amplifiers in circuits with electron tubes and semiconducting devices," spoke of the modernization of magnetic amplifiers carried out for the purpose of using them in circuits for computing machines, and also of attempts to create ac bridge magnetic amplifiers with triode excitation and motors at the outputs.

Ia. M. Kleiman and L. S. Koniaeva, in their paper, "Circuits with magnetic amplifiers for the control of the main drive of reversible roller motors," presented interim results of the work on creating circuits for controlling main drives of blooming machines with magnetic amplifiers as pre-amplifiers to electromagnetic amplifiers.

The representative of the Chinese Academy of Science, Professor Ming Nai-ta, presented a paper devoted to the synthesis of electric circuits with dissipative elements.

Ia. B. Rozman, in the paper "Magnetic amplifiers in regulated dc electric drives for machines," dealt with the development and installation of dc motor regulation with application made of magnetic amplifiers and semiconductor-magnetic amplifiers, and of their advantages and disadvantages, and provided results of the testing of foreign regulated drives by magnetic amplifiers.

The paper of Ia. N. Shtrafun, "The use of high-power chokes and transformers with magnetization in excitation systems of power turbogenerators, " was devoted to the development of a number of excitation systems using contactless magnetic elements, and also to high-power semiconductor rectifiers. The speaker related that the results of field investigation of the new turbogenerator excitation system at powers of 30,000 kilowatts showed a sufficiently high quality and demonstrated the efficacy of using it for series high-power turbogenerators.

E. F. Stepura and V. V. Semenov, in the paper "Some applications of saturated chokes in electromechanical automatic control systems," dealt with work on the creation of contactless nonlinear and computing elements for use in electromechanical automatic control systems for the purpose of improving their dynamic characteristics. Such elements possess simplicity and high reliability, and their errors do not exceed 2-5%.

V. G. Komar and P. P. Mokhov gave a paper on "High-power static ac voltage stabilizers with magnetic amplifiers," in which they dealt with developments of high-power three-phase voltage stabilizers at 300 and 500 kilowatts for searchlight filament sources. In the paper a design was given for connecting saturated chokes, guaranteeing a high-power coefficient and an increased coefficient of efficiency.

F. F. Sokolov, in the paper "A single-phase voltage stabilizer with saturated chokes for powers of 1.25 kilowatts," spoke of the selection of the sensing, amplifying and executive organs of the stabilizer, and gave a method for determining the computed power of various stabilizers in order to compare them.

B. A. Mitrofanov and N. I. Chicherin gave a paper on "Servosystems, ac, using magnetic amplifiers and ac motors with powers of 0.5-1 kilowatt," devoted to the development of ac servo drives (two-phase and three-phase), and giving the basic results of the work cited.

S. S. Roizen, in his paper, "Rolling mill electric drives with magnetic amplifiers," dealt with certain cases of the use of magnetic amplifiers in rolling mill electric drives. The speaker cited the efficacy of use of three-phase two-stage designs for the output magnetic amplifier cascades, and also recommended the use of higher-frequency magnetic amplifier supplies for the improvement of the dynamic characteristics of the electric drives. An example was given in the paper of the use of magnetic amplifiers in the electric drives of a reduction turbo-rolling mill containing twenty stands, each with an individual dc motor.

B. Section on Discrete Magnetic Elements

A. I. Bakhir, in the paper "Durable and operative memory devices of ferrites and semiconducting elements," considered the over-all design for constructing a device in which the writing of signs is carried out by means of

leads drawn through ferrites according to a given rule, while for choosing the address and reading the information written are used decoders and shapers, implemented completely by ferrites and semiconducting elements.

In the paper of Mr. Ovcharenko, "Basic logical circuits, executed by magnetic elements and transistors," there were considered the different characteristics of computing cells, made up of ferrites and transistors, and various types of such logical circuits were given.

Ia. A. Khetagurov, in the paper "On one design for filling devices made of ferrites," dealt with the design of a filling device which might be used in hysteresis machines, and with the peculiarities of developing such designs and its basic blocks. The speaker gave a detailed treatment of the frequency characteristics of the design, and gave results of tests of the design with various types of ferrites, and also gave data on a small-size memory device executed with semiconductors.

In the paper of K. G. Mitiushkin and V. A. Zhozhikashvili, "The use of contactless magnetic elements in tele-control devices," there was noted the superiority of contactless magnetic apparatus for the construction of tele-devices by the method of continuous transmission of communications, and two types of contactless devices for tele-control were characterized: time – the distributing device works sporadically – and continuously acting devices with a pulse distributor source from a common system with synchronous frequency.

V. A. Ordynets, in the paper "Computing cells of magnetic elements," spoke of a series of circuits constructed of ferrite cores, in particular, of circuits of the choke type, and of circuits using the saturation effect of the individual portions of the magnetic conductors, and gave practical circuits for registers, adders, triggers, valves, etc.

The paper of N. M. Brusentsov, "Digital elements of the type of magnetic amplifiers with a source of current pulses," was devoted to the use of such amplifiers as elements of high-speed digital machines. The author considered the advantages, disadvantages and characteristics of elements of this type, gave a method for designing such elements, and gave examples of circuits for ternary algebraic adders, ternary inverters, ternary decoders and switches.

In the paper of E. F. Berezhnii, "Magnetostrictive elements in computing technology devices," there were considered the design and construction of magnetostrictive delay lines for memory devices. The author provided results of an investigation into the influence of temperature on the stability of the delay time and also answered questions on compensation for the variation in delay time due to temperature. The characteristics were given for operative registers composed of magnetostrictive delay lines.

In the paper of V. V. Bardizh , "Questions of pulse magnetization of magnetic cores," the working conditions were investigated of ferrite cores with rectangular hysteresis loops in matrix-type, magnetically operating memory devices, and the basic requirements for increasing the speed of action of devices of this type were formulated.

E. I. Gurvich gave a paper on "Pulse magnetization of soft magnetic alloys with rectangular hysteresis loops," in which the dependence of magnetization time on pulse current amplitude was considered, and in which experimental data were given on the switching (magnetization) time of cores prepared from strips of various thicknesses.

In his communication, "Experimental investigation of commutators implemented by magnetic elements with rectangular hysteresis loops," V. N. Tutevich provided experimental data from the investigation of commutators used in remote control systems, and characterized the influence exerted by the form of the supply pulses on the effectiveness of commutator functioning.

The communication of V. V. Verigin, N. S. Kartseva, and S. P. Maslov, "Matrix memory devices," was devoted to the principles and peculiarities of performance of ternary memory devices. The authors dealt with the functioning of blocks of devices of this type.

A. S. Fedorov gave a communication on, "Increasing the speed of action of ferrite core memory devices," in which he cited several methods of increasing both the speed of action and the reliability of matrix type memory devices. The author based his choices on the optimal relationships between speed and capacity of the memory devices, and he provided experimental results of memory device testing.

In the paper of M. P. Sycheva, "Certain questions pertaining to increasing the reliability of magnetically operating memory devices," there was investigated the ferrite memory device used in the BESM computer, and comparisons were made between various magnetic materials used for memory device cores. Also given were results of experimental investigations of models of memory devices.

V. A. Zhozhikashvili and K. G. Mitiushkin gave a paper "On a method of designing pulse distributors (commutators) made of magnetic elements with rectangular hysteresis loops." The design is done by the method of obtaining the minimum power necessary for transmitting information from one element to another, taking into account the load power. The design method allows one to find the optimal distributor parameters for the most economic expenditure of energy in the distributor.

In the paper "Relay phenomena in ring circuits containing magnetic cores with rectangular hysteresis loops," the same authors considered the questions of stability and transient response in ring circuits.

V. V. Kobelev and Iu. I. Vizun, in the paper "Program-controlled magnetic memory devices," reported on a shift register circuit of ferrite-transistor cells, using plane crystal triodes, which made it possible to obtain currents sufficient for the reading and writing of information in the memory devices.

Iu. S. Volkov presented a paper, "Inductive transformer of angular shaft rotations into digits, expressed in the binary system in the form of pulse sequences." The transformer gives shaft rotations with an accuracy of three minutes of arc in the form of thirteen-place binary numbers.

V. A. Ziminym, in the paper "Shaping transformers," showed that the use of transformers for pulse shaping provides for stable machine functioning, and also for simplicity of the circuits. The author gave a circuit for a conversion transformer and a mathematical analysis of its functioning. Formulas were derived for designing the shaping transformer; experimental data were given.

Ia. G. Koblents,in the paper "Construction of devices for the automatic control of an ATS (Automatic Telephone Station) based on contactless magnetic elements with rectangular hysteresis loops," considered the requirements on commutation elements, the construction of elementary operational circuits, registers, random access circuits, subscriber assemblies, circuits for recognizing the number of the calling subscriber and transmitting information along communication lines.

V. P. Sabadashev, in the paper " Contactless tele-control devices with phase selecting indications," gave an analysis of phase-sensitive circuits with saturated transformers, obtained their loading characteristics analytically, and gave experimental data and recommendations on the application of various remote control circuits with phase selecting indications.

In the paper "On the construction of circuit elements not requiring materials with rectangular hysteresis loops," V. A. Leokene dealt with the use of semiconducting diodes with large nonconducting portions (insensitivity threshold) for the construction of economical and operative computing cells from materials without rectangular hysteresis loops.

V. P. Sal'nikova, in the paper "Technological problems of the preparation of ferrite cores with rectangular hysteresis loops," reported on experiments and research concerning the development of new types of ferrites and their application to the mass production of ferrite cores.

The paper of Eng. Iu. Ia. Dusavitskii, on the subject, "Applications of contactless magnetic relays for the automatic control of furnaces for metal heat-treating," was devoted to the use of contactless magnetic relays as measuring organs in automatic current control circuits. The paper included information on the method of selecting the block schematic of the device to be regulated, and provided design and experimental data on the magnetic amplifiers used.

In his paper, "On magnetic memory devices," F. V. Maiorov considered several circuits for controlling magnetic memory devices by means of semiconducting elements and ferrite cores: circuits for reading and writing amplifiers using transistors, circuits for transistor diode matrices taking coded numbers from a simple diode decoder for the coded address, writing and reading circuits of crystal triodes, and reading register circuits, also containing semiconducting triodes. The peculiarities of functioning, and basic data, were considered for the circuits cited.

In the paper of M. I. Sataev, "Transformation of shaft rotation to a number and the inverse transformation of a number to a shaft rotation," the author gave the principles of transforming shaft rotations to binary numbers in a digital machine by means of miniature transformers constructed of semiconductors and ferrites, and also described miniature inverse transformers and gave the basic characteristics of these devices.

L. I. Gutenmakher presented a paper, "Work of the Electro-Modelling Laboratory (Laboratorii elektromodelirovaniia) in the domain of creating contactless elements and systems." The basic subject of the Laboratory's work is the creation of information machines with large, fast-acting memories with the purpose of simulating complex processes of mental work. In the paper there was considered the design of an information machine and the specific properties of its elements, in particular, new types of economic address systems. The use of operational ferrite memory systems allows the machine dimensions to be decreased. There was developed in the Laboratory a universal digital machine without mechanical moving parts, the memory device of which was implemented by ferrites controlled by magnetic elements without recourse to vacuum tubes.

In the paper of I. B. Negnevitskii and R. A. Lipman, "Transient responses in magnetic amplifiers in the relay regimen," there was given the physical basis and a quantitative analysis of the delay phenomena in operating and releasing a contactless relay. It was shown that the lag in load current variation occurs until the core induction attains the saturation induction value; thereafter, the ordinary transient response occurs, in accordance with the relay type. Expressions were obtained for computing the total times of operation and release of magnetic relays.

N. M. Tishchenko, in the paper "Contactless relay times," dealt with the development of one type of contactless relay, and gave its design and experimental data.

The work of the Electro-Modelling Laboratory in the application of ferrites with rectangular hysteresis loops was detailed in the paper of A. A. Kosarev. The author noted that a disadvantage of existing ferrite-diode designs is the necessity of using diodes which were not specially designed to work with very small ferrites, as a result of which multi-turn windings are needed, thus reducing the speed of action of the design. Diode-less variations of the device were considered, in particular, transfluxors. However, as the author noted, ferrite-diode designs are today more efficient, and are recommended to a higher degree, than are ferrite-transistors.

N. V. Korol'kov gave a short communication on a two-core trigger developed at the Electro-Modelling Laboratory.

E. Dzhakov, Corresponding Member of the Bulgarian Academy of Science, considered, in his paper, the times of relays with magnetic amplifiers having delay times of 10 seconds and higher. The delay time of the relay is attributable to an RC network at the magnetic amplifier input. An analysis was given in the paper of the influence of the various circuit parameters on the delay time, and relationships were given for choosing the designs of the magnetic amplifiers used in the given relay.

At the conclusion of the plenary session, E. T. Chernyshev, N. G. Chernysheva, and E. N. Chechurina gave a paper on the subject "Contemporary state of questions on the testing of magnetic materials in dynamic regimens." The authors considered methods of obtaining magnetic characteristics with an estimate of the possible errors of measurement.

A paper on the subject "Fundamental principles of constructing series toroidal cores for magnetic amplifiers and contactless elements of automation," was presented by O. A. Sedykh and M. A. Rozenblat. Presented in the paper were the fundamental principles of constructing series toroidal cores for magnetic amplifiers and contactless magnetic elements for automation, and there was given a method for selecting the various core dimensions. The authors gave a design for series toroidal cores recommended for line production, meeting the basic requirements of industry for toroidal cores for magnetic amplifiers and contactless magnetic elements by means of the least possible number of standard-dimension mass-produced cores.

V. E. Pankin gave a communication on the results of development of series standard toroidal cores for magnetic amplifiers.

The representative of the Czechoslovakian Academy of Science, Iu. Gashkovets, in his paper, gave a detailed discussion of the development of magnetic amplifiers in Czechoslovakia.

The representative of the Polish Academy of Science, S. Vendzhin, spoke of the use, in Poland, of magnetic amplifiers in industrial automation devices.

In conclusion, the participants at the Conference adopted a resolution in which it was noted that magnetic elements are very promising elements of automation, remote control and computing technology, and a series of measures was adopted directed toward improving research work in the domain of magnetic elements and their most intensive injection into industry.

M. A. Boiarchenkov, V. S. Volodin, F. I. Kerbnikov,
G. D. Kozlov, G. V. Subbotina, and I. S. Trefilova.